Ring-defining p-adic valuations

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The p-adics

Consider the following formula, due to Julia Robinson:

$$\varphi(x) : \exists y\ 1 + px^2 = y^2$$

We claim that $\varphi(\mathbb{Q}_p) = \{x \in \mathbb{Q}_p \mid \varphi(x)\} = \mathcal{O}_{v_p}$. Indeed:

- If $x \notin \mathcal{O}_{v_p}$ then $v_p(1 + px^2) = v_p(px^2) = 2v_p(x) + 1$ and is odd;
- If $x \in \mathcal{O}_{v_p}$ then $v_p(px^2) > 0$ and $X^2 - (1 + px^2)$ has a root by Hensel's lemma.

For a valued field (K, v), we say that v is *ring-definable* if there is a formula such that $\varphi(K) = \mathcal{O}_v$.

Nice extensions of the p-adics

Let $K/\mathbb{Q}_p$ be algebraic. We have:

$$\mathbb{Z} \subseteq v_pK \subseteq \mathbb{Q} \quad \& \quad \mathbb{F}_p \subseteq Kv_p \subseteq \mathbb{F}_p^{\text{alg}}$$

Extensions are *nice* when either $v_pK \neq \mathbb{Q}$ or $Kv_p \neq \mathbb{F}_p^{\text{alg}}$. In nice extensions, v_p is again ring-definable:

If $v_pK \neq \mathbb{Q}$

Take $t \in K$ such that $v(t) = \gamma > 0$ and is not q-divisible for some prime q. The following set is ring-definable:

$$I = \{x \in K \mid \exists y\ 1 + tx^q = y^q\} = \{x \in K \mid \gamma + qv(x) > 0\}$$

It is not quite $\mathcal{O}_{v_p}$ but it contains it. Consider its stabilisator:

$$R = \{a \in K \mid aI \subseteq I\}$$

R is a ring and contains $\mathcal{O}_{v_p}$, it is therefore a coarsening of it; it is non-trivial since $t^{-2} \notin R$. The only possibility is $R = \mathcal{O}_{v_p}$, which is thus ring-definable.

If $Kv_p \neq \mathbb{F}_p^{\text{alg}}$

We take a polynomial f such that $\overline{f}$ has no root and $\overline{f}'$ is not zero. We obtain a ring-definable set:

$$\mathcal{M}_{v_p} \subseteq \frac{1}{f(K)} - \frac{1}{f(K)} \subseteq \mathcal{O}_{v_p}$$

In order to obtain $\mathcal{O}_{v_p}$ we need to add a ring-definable set T which contains a lift of every element of Kv_p. If the latter is finite, we can just take lifts of its element as parameters. If it is infinite, then it is PAC, and we can add the following set:

$$T = \frac{1}{f(K)} \cdot \frac{1}{f(K)} \text{ is such that } \overline{T} \supseteq Kv_p$$

In these definitions, we allow parameters and we do not control quantifiers. More careful constructions can be done, for example in [3].

Wild extensions of the p-adics

Both previous definitions fail when $v_pK = \mathbb{Q}$ and $Kv_p = \mathbb{F}_p^{\text{alg}}$. When $K = \mathbb{Q}_p^{\text{alg}}$, we know by minimality of algebraically closed fields that no definition can exist; however the defect of mixed characteristic fields means that the case $K \neq \mathbb{Q}_p^{\text{alg}}$, $v_pK = \mathbb{Q}$ and $Kv_p = \mathbb{F}_p^{\text{alg}}$ does occur. These are the *wild* extensions of $\mathbb{Q}_p$, for which no explicit definition is known; yet we can still show that v_p is ring-definable.

p-henselianity

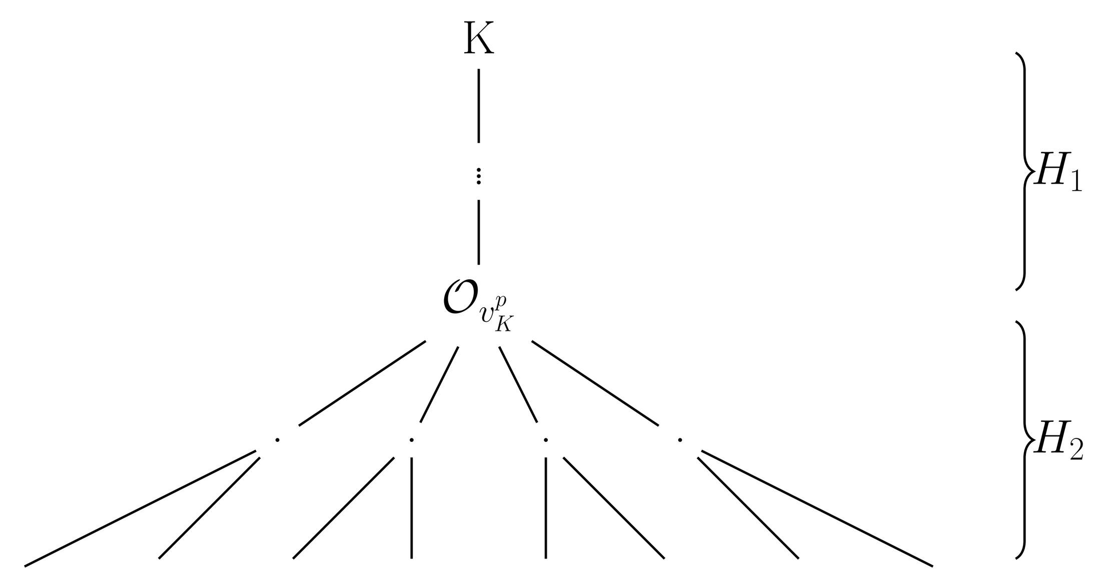
A valuation v on a field K is called *p-henselian* if it extends uniquely to the p-closure $K(p)$, which is the compositum of all Galois extensions of K of p-power degree. One can prove that if a valuation extends uniquely to every Galois extension of degree p, then it is already p-henselian: this is achieved by using Galois theory. See for example [2].

The tree structure

p-henselian valuation rings are well-behaved regarding inclusion, forming a tree structure with 2 meaningful components:

$$H_1 = \{\mathcal{O}_v\ p\text{-henselian} \mid Kv \neq Kv(p)\}$$
$$H_2 = \{\mathcal{O}_v\ p\text{-henselian} \mid Kv = Kv(p)\}$$

H_1 is linearly ordered, and every ring of H_2 is included in every ring of H_1.



v_K^p: The canonical p-henselian valuation

In the middle of the tree lies one ring, the valuation of which we denote by v_K^p. It is the *canonical p-henselian* valuation, and it is characterized by the following properties:

- It is comparable with every p-henselian valuation ring,
- Every proper coarsening of it has non p-closed residue field,
- Every proper refinement of it has p-closed residue field,
- It is trivial iff K is p-closed or K has no non-trivial p-henselian valuation.

Main theorem (Jahnke-Koenigsmann, 2015) [4]

When $p \neq 2$, if K is a field of characteristic p or if K contains a primitive p^{th}-root of unity, then v_K^p is $\emptyset$-ring-definable.
When $p = 2$, another valuation called v_K^{2*} is $\emptyset$-ring-definable, and in the case where its residue field is non-euclidean, we have $v_K^{2*} = v_K^2$.

This result is obtained by cleverly stating "$\mathcal{O} = \mathcal{O}_{v_K^p}$" in first-order in the language of rings augmented with a predicate for $\mathcal{O}$, and then applying Beth's theorem; this definability is therefore in no way explicit. The assumptions on K are present in order to control the Galois extensions of degree p, which will be either Artin-Schreier or Kummer extensions.

What is $v_{\mathbb{Q}_b}^p$?

Since v_b is henselian it is in particular p-henselian for any p. It must therefore be comparable with the canonical p-henselian valuation, and we have to look at two cases:

- If $\mathcal{O}_{v_b} \subseteq \mathcal{O}_{v_{\mathbb{Q}_b}^p}$, then there must be a convex subgroup of $v_b\mathbb{Q}_b$ corresponding to this coarsening; but since $v_b\mathbb{Q}_b = \mathbb{Z}$, the only possibility is $\mathcal{O}_{v_{\mathbb{Q}_b}^p} = \mathcal{O}_{v_b}$.
- If $\mathcal{O}_{v_{\mathbb{Q}_b}^p} \subseteq \mathcal{O}_{v_b}$, then $\mathcal{O}_{v_{\mathbb{Q}_b}^p}/\mathcal{M}_{v_b}$ is a valuation ring of $\mathbb{Q}_b v_b = \mathbb{F}_b$, which has no non-trivial valuation, so again $\mathcal{O}_{v_{\mathbb{Q}_b}^p} = \mathcal{O}_{v_b}$.

The argument works in the same manner for a non p-closed algebraic extension K of $\mathbb{Q}_b$, since $\mathbb{Z} \subseteq v_bK \subseteq \mathbb{Q}$ has no non-trivial convex subgroup, and $\mathbb{F}_b \subseteq Kv_b \subseteq \mathbb{F}_b^{\text{alg}}$ has no non-trivial valuation.

There and back again

Let $\mathbb{Q}_b \subseteq K \subsetneq \mathbb{Q}_b^{\text{alg}}$. We have to go to a non p-closed extension which contains a primitive p^{th}-root of unity, and then back to K by interpretability:

- $K \neq K^{\text{alg}}$, thus there exists a finite algebraic extension of K of degree $n \geqslant 2$, which can be extended to a Galois extension M of degree at most $n!$.
- Let p divide $[M : K]$, then $\text{Gal}(M/K)$ has a p-Sylow subgroup S_p; denote F its fixed field. Now M/F is a Galois extension of p-power degree, therefore F is not p-closed, and F/K is finite.
- Consider $L = F[e^{\frac{2\pi i}{p}}]$, L is still not p-closed since it is a finite extension of F, so v_b is definable in L.
- Finally, we interpret L in K (with coefficients of minimal polynomials of generators of L as parameters), and the restriction of v_b to K is therefore definable.

An application: NIPity in extensions of $\mathbb{Q}_p$

$\mathbb{Q}_p$ is NIP, but are its algebraic extensions all NIP? Ring-defining the valuation tells us that K is NIP iff (K, v) is NIP. Now, by interpretability, (K, v) NIP implies Kv NIP. But if we take $K/\mathbb{Q}_p$ algebraic with infinite but not separably closed residue field, Kv_p is PAC, and thus has IP [1]:

$$Kv_p \text{ infinite not SC } \Rightarrow Kv_p \text{ has IP } \Rightarrow K \text{ has IP}$$

Therefore not all algebraic extensions of $\mathbb{Q}_p$ are NIP.

References

[1] J.-L. Duret. Les corps faiblement algébriquement clos non separablement clos ont la propriété d'indépendance. 1980.

[2] A. J. Engler and A. Prestel. *Valued fields*. Springer, 2010.

[3] A. Fehm. Existential $\emptyset$-definability of henselian valuation rings. *The Journal of Symbolic Logic*, 80(1):301–307, 2015.

[4] F. Jahnke and J. Koenigsmann. Uniformly defining p-henselian valuations. *Annals of Pure and Applied Logic*, 166, 07 2014.