Definability of types, symmetry and stationarity in local stability

Seminar: Introduction to stable theories

Blaise Boissonneau

January 10, 2022

Source The content of this talk is adapted from Daniel Palacín's Chapter in *Lectures in Model Theory*, EMS, 2018:
`https://www.ems-ph.org/books/book.php?proj_nr=229`

Conventions We fix a complete $\mathcal{L}$-theory T and work in a Monster model $\mathcal{M}$. We write single letters a, b, $c\ldots$ for (finite) tuples of elements and x, y, $z\ldots$ for tuples of variables. For a partitionned formula $\varphi(x, y)$, we let $\varphi * (y, x) = \varphi(x, y)$. We write *we* for *Many cups of coffee, the very same joke every time and I.*

Stone spaces

Recall that:

- A basis of open sets for $S_n(A)$ is the collection of $[\psi] = \{p \in S_n(A) \mid \psi \in p\}$ for $\psi \in \mathcal{L}(A)$. Each $[\psi]$ is also closed and $S_n(A)$ is compact.
- Similarily, when $\psi(x)$ is a Boolean combination of $\varphi(x, a)$ and $\neg\varphi(x, a')$ (for $a, a' \in A$), $[\psi] = \{p \in S_\varphi(A) \mid p \vdash \psi\}$ is a basic open set of $S_\varphi(A)$.
- Let $\pi_\varphi : S_n(A) \to S_\varphi(A); p \to p|_\varphi$. It is a continuous function, thus $S_\varphi(A)$ is also compact.

1 Definability of types

We always assume that $\varphi(x, y)$ is stable.

Lemma 1. *For any $p(x) \in S_n(A)$, there is $q \in S_\varphi(\mathcal{M})$ consistent with p and $\mathrm{acl}^{eq}(A)$-definable.*

Proof. Let $X_0 = \{q \in S_\varphi(\mathcal{M}) \mid p \cup q \text{ is consistent}\}$. Note that $X_0 = \pi_\varphi[p]$ where $[p] = \bigcap_{\psi\in p}[\psi]$; in particular, X_0 is closed.

For each i, define X_{i+1} to be the set of accumulation points of X_i:

$$X_{i+1} = \{p \in X_i \mid \forall\psi \in \mathcal{L}|_\varphi(\mathcal{M}),\, p \in [\psi] \Rightarrow \exists p' \in X_i \cap [\psi], p \neq p'\}$$

Any $p \in X_i \setminus X_{i+1}$ is isolated by a formula, thus all X_i are closed.

Suppose X_{n+1} is non-empty:

- There is $\psi \in \mathcal{L}|_\varphi(A)$ and $p, p' \in X_n$ distinct but both implying ψ.
- p and p' are distinct, so for some a, $\varphi(x, a) \in p$ and $\neg\varphi(x, a) \in p'$.
- $\psi \wedge (\neg)\varphi(x, a)$ is a Boolean combination of φ, and since $p \vdash \psi \wedge \varphi(x, a)$, we can find $p'' \in X_{n-1}$ distinct also implying it.

In the end, we obtain a binary tree of formulas of height n. Thus, by compactness, if X_n is never empty, we can find a binary tree of any height. Take μ such that $|2^{<\mu}| \leqslant |T| < 2^\mu$, and consider a binary tree of height μ – thus having 2^μ many branches and $|2^{<\mu}|$ many formulas. We can find a model $\mathcal{N}$ of size $|T|$ containing all parameters of this tree. Now:

- $S_\varphi(\mathcal{N}) \geqslant 2^\mu > |T|$ because each branch can be extended into at least one φ-type,
- $S_\varphi(\mathcal{N}) \leqslant |T|$ because φ-types over models are definable (recall that φ is stable).

Hence, there must be some smallest n for which $n + 1$ is empty (clearly X_0 is non-emtpy). This means that any $p \in X_n$ is isolated by some formula ψ_p. Because $X_n = \bigcup_{p \in X_n} [\psi_p]$ and since X_n is compact, it must be finite.

Take $q \in X_n$, it is definable (seen last week); but since X_n is A-invariant, any conjugate of q over A must lie in X_n, so there are finitely many, and the canonical basis of q must be in $\mathrm{acl}^{eq}(A)$. □

2 Symmetry

Lemma 2 (Harrington)**.**

$$d_p x \varphi(x, y) \in q(y) \Leftrightarrow d_q y \varphi(x, y) \in p(x)$$

Proof. Definitions of φ in p and q exist because φ is stable. Let A be parameters of both definitions. We let b_0 realize $q|_A$ and a_0 realize $p|_{A \cup \{b_0\}}$. By induction, we let b_n realize $q|_{A \cup \{a_0, \cdots, a_{n-1}\}}$ and a_n realize $p|_{A \cup \{b_0, \cdots, b_n\}}$. Thus:

- For $i \geqslant j$, $\vDash \varphi(a_i, b_j)$ iff $\varphi(x, b_j) \in p$ iff $\vDash d_p x \varphi(x, b_j)$ iff $d_p x \varphi(x, y) \in q$,
- For $i < j$, $\vDash \varphi(a_i, b_j)$ iff $\varphi(a_i, y) \in q$ iff $\vDash d_q y \varphi(a_i, y)$ iff $d_q y \varphi(x, y) \in p$.

φ is stable, thus there must be $i < j$ and $i' \geqslant j'$ such that $\vDash \varphi(a_i, b_j) \leftrightarrow \varphi(a_{i'}, b_{j'})$. □

3 Stationarity

We call $S_{\varphi^+}(A)$ the set of complete generalized φ-types: they are allowed to contain formulas $\psi(x,a)$ with $a \in A$ equivalent to boolean combinations of φ with parameters anywhere. Over models, generalized φ-types are the same as φ-types.

Lemma 3. *For any generalized φ-type over $A = \mathrm{acl}^{eq}(A)$, there is a unique A-definable type extending it to $\mathcal{M}$.*

Proof. Let $A = \mathrm{acl}^{eq}(A)$ and $p \in S_{\varphi^+}(A)$. By the first lemma, there is a type in $S_\varphi(\mathcal{M})$ extending p which is definable over A.

Now let $p_1, p_2 \in S_\varphi(\mathcal{M})$ extending p be A-definable. Take $b \in \mathcal{M}$, we aim to prove that $\varphi(x,b) \in p_1$ iff $\varphi(x,b) \in p_2$.

- By the first lemma, there is $q(y) \in S_{\varphi^*}(\mathcal{M})$ definable over A and consistent with $\mathrm{tp}(b/A)$.
- For completions p_1', p_2' and q' of p_1, p_2 and $q \cup \mathrm{tp}(b/A)$ we can apply Harrington's Lemma:

$$\varphi(x,b) \in p_i \Leftrightarrow d_{p_i'} x\varphi(x,y) \in \mathrm{tp}(b/A) \subset q' \Leftrightarrow d_{q'} y\varphi(x,y) \in p_i'$$

- We know (see last week) that $d_{q'}y\varphi(x,y)$ is equivalent to a positive Boolean combination of $\varphi(a,y)$.
- We also know it has parameters in A, thus p knows about it:

$$d_{q'}y\varphi(x,y) \in p_i' \Leftrightarrow d_{q'}y\varphi(x,y) \in p$$

Hence $p_1 = p_2$ and we have uniqueness. Note that we only use acl^{eq}-closure in the last step. □

Lemma 4. *Let $p \in S_\varphi(\mathcal{M})$ be definable over a model $\mathcal{N}$ and consistent with a partial type $\pi(x)$ over $\mathcal{N}$; they are finitely co-satisfiable in $\mathcal{N}$.*

Proof. Clearly the restriction of p to $\mathcal{N}$ is finitely co-satisfiable with π in $\mathcal{N}$; hence there is $q \in S_x(\mathcal{M})$ extending $p \cup \pi$ finitely satisfiable in $\mathcal{N}$. Its restriction $q|_\varphi$ is thus definable over $\mathcal{N}$ (see last week).

We show $q|_\varphi = p$. Assume not; then there is $c \in \mathcal{M}$ such that $\varphi(x,c) \in p$ and $\neg\varphi(x,c) \in q$. This means:

$$\begin{aligned}
\mathcal{M} &\vDash d_p x\varphi(x,c) \wedge \neg d_q x\varphi(x,c) \\
\mathcal{M} &\vDash \exists y(d_p x\varphi(x,y) \wedge \neg d_q x\varphi(x,y)) \\
\mathcal{N} &\vDash \exists y(d_p x\varphi(x,y) \wedge \neg d_q x\varphi(x,y))
\end{aligned}$$

But that can't be because p and q agree for φ on $\mathcal{N}$. □

4 Dividing

Proposition 5. *Fix a and A, TFAE:*

1. $\varphi(x,a)$ *is satisfiable in every model containing A*
2. $\varphi(x,a)$ *doesn't fork over any model containing A*
3. $\varphi(x,a)$ *doesn't divide over A*
4. *There is a formula $\chi(x) \in \mathcal{L}(A)$ equivalent to a positive Boolean combination of A-conjugates of $\varphi(x,a)$*
5. *There is $p \in S_\varphi(\mathcal{M})$ $\mathrm{acl}^{eq}(A)$-definable and containing $\varphi(x,a)$.*

The most important equivalence here is between 3 and 5, at least for this talk.

Proof.

1⇒2 Assume $\varphi(x,a) \vdash \bigvee_{i<n} \varphi_i(x,b_i)$. Let $d \in \mathcal{N}$ realize $\varphi(x,a)$, then there must be i such that $\mathcal{N} \vDash \varphi_i(d,b_i)$. But this also holds for any $b' \equiv_{\mathcal{N}} b$.

2⇒3 We saw last week that a formula divides over a set iff it divides over some model containing this set.

5⇒1 A type $\mathrm{acl}^{eq}(A)$-definable is also definable over any model containing A, and thus by the previous lemma, finitely satisfiable in this model.

4⇒5 By our first lemma, there is $p \in S_\varphi(\mathcal{M})$ $\mathrm{acl}^{eq}(A)$-definable and consistent with $\chi(x)$. Hence one of the formulas $\varphi(x,a')$ appearing in $\chi(x)$ must lie in p. Let $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$ send a' to a and consider σp.

3⇒4 Let $p = \mathrm{tp}(a/A)$. By our first lemma, there is a type $q \in S_{\varphi^*}(\mathcal{M})$ consistent with p and $\mathrm{acl}^{eq}(A)$-definable, and thus definable in some model $\mathcal{N}$ containing A. By the previous lemma, $p \cup q$ is finitely satisfiable in $\mathcal{N}$, and we can find $q' \in S_n(\mathcal{M})$ finitely satisfiable in $\mathcal{N}$ containing p and q.

Since φ^* is stable, $q'|_{\varphi^*}$ – which is exactly q – is definable by a positive Boolean combination of $\varphi(x,c_i)$, where each c_i realizes $q|_{\mathcal{N}\cup\{c_0,\cdots,c_{i-1}\}}$. Because q' is A-invariant (by finite satsfiability in $\mathcal{N}$), we can take $(c_i)_{i<\omega}$ $\mathcal{N}$-indiscernible, thus also A-indiscernible. Furthermore, q contains $\mathrm{tp}(a/A)$, so $c_i \equiv_A a$. Because $\varphi(x,a)$ doesn't divide, the definition of $q'|_{\varphi^*}$ is satisfiable.

Recall that q was taken to be $\mathrm{acl}^{eq}(A)$-def: there is $\psi(x,d)$ with $d \in \mathrm{acl}^{eq}(A)$ defining q, hence equivalent to the positive Boolean combination of $\varphi(x,c_i)$ defining $q'|_{\varphi^*}$. Now consider $\chi(x) = \bigvee_{d \equiv_A d'} \psi(x,d')$. □

We now consider several formulas at once. The following result is just a coding trick:

Lemma 6. *Let $\Delta(x, y)$ be a finite set of formulas and let $n = |\Delta|$. There is $\chi_\Delta(x; y_0, \cdots, y_n, z, z_0, \cdots, z_{2n})$ such that:*

- *If A has a least 2 elements, for $a \in A$ and $\varphi \in \Delta$, there are $b, b' \in A$ such that $\varphi(x, a) \leftrightarrow \chi_\Delta(x, b)$ and $\neg\varphi(x, a) \leftrightarrow \chi_\Delta(x, b')$.*
- *Given $b \in A$ such that $\psi_\Delta(x, b)$ is consistent, there is $a \in A$ and $\varphi \in \Delta$ such that $\chi_\Delta(x, b) \leftrightarrow \varphi(x, a)$ or $\chi_\Delta(x, b) \leftrightarrow \neg\varphi(x, a)$.*
- *χ_Δ is stable iff all formulas in Δ are stable.*

Thus we can think about Δ-types as χ_Δ-types.

Proposition 7. *Let $\varphi(x, y)$ and $\psi(x, z)$ be stable. If both $\varphi(x, a)$ and $\psi(x, b)$ divide over A, so does $\varphi \vee \psi(x; a, b)$.*

In particular, for any a and A, we obtain that either $\varphi(x, a)$ or $\neg\varphi(x, a)$ does not divide over A.

Clearly this doesn't work for unstable formulas (recall the cyclical order example).

The strategy of the proof is as follows: $\varphi \vee \psi$ does not divide iff it is contained in an $\mathrm{acl}^{eq}(A)$-definable $\varphi \vee \psi$-type. Then this type – or rather, another type very similar but allowed to contained more formulas – must contain φ or ψ, thus φ (or ψ) is contained in an $\mathrm{acl}^{eq}(A)$-definable φ (or ψ)-type, which is equivalent to φ (or ψ) not dividing.

Proof. Let $\Delta = \{\varphi, \psi, \varphi \vee \psi\}$ and let χ_Δ encode it as above. χ_Δ is stable. The formula $\varphi \vee \psi(x; ab)$ does not divide over A iff there is $p \in S_{\varphi\vee\psi}(\mathcal{M})$ $\mathrm{acl}^{eq}(A)$-definable and containing it.

Let $p|_{\mathrm{acl}^{eq}(A)^+} \in S_{\varphi\vee\psi^+}(A)$ restrict p. By our first lemma, there is $q \in S_{\chi_\Delta}(\mathcal{M})$ $\mathrm{acl}^{eq}(A)$-definable and consistent with $p|_{\mathrm{acl}^{eq}(A)^+}$.

There is a $q' \in S_\Delta(\mathcal{M})$ equivalent to q; hence q' is A-invariant and thus $q'|_{\varphi\vee\psi}$ is $\mathrm{acl}^{eq}(A)$-definable.

p and $q'|_{\varphi\vee\psi}$ are two $\mathrm{acl}^{eq}(A)$-definable extensions of $p|_{\mathrm{acl}^{eq}(A)^+}$, thus they are equal and $\varphi \vee \psi(x; ab) \in q'$. This yields that either $\varphi(x, a)$ or $\psi(x, b) \in q'$.

Both $q'|_\varphi$ and $q'|_\psi$ are also $\mathrm{acl}^{eq}(A)$-definable and thus the formula contained in q' can't divide over A. □

As a direct corollary, we get that in a stable theory, dividing and forking are equivalent.