Definable and algebraic closures

Seminar on continuous logic

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The content of this talk is adapted from I. Ben Yaacov, A. Berenstein, C. Ward Henson & A. Usvyatsov, *Model theory for metric structures*, sections 9-10.

convention When not stated otherwise, $\mathcal{M}$ is always a (complete) metric structure with signature $\mathcal{L}$, and A is a subset of M. We let letters like x denote variables or tuples of variables, and we let sup or inf run over tuples. We write *we* for *many cups of coffee, the induced lack of sleep and I.*

Definability

Recall that in continuous logic:

- A predicate $P : M^n \to [0, 1]$ is definable over A if it is a uniform limit of $\mathcal{L}(A)$-formulas;
- a set $D \subseteq M^n$ is definable if $P(x) = \mathrm{dist}(x, D)$ is a definable predicate;
- a function $f : M^n \to M$ is definable if $P(x, y) = d(f(x), y)$ is a definable predicate.

When $\mathcal{M}$ is sufficiently saturated, functions are definable iff their graphs are type-definable:

Proposition 1. *If $\mathcal{M}$ is $|A|^+$-saturated, then a function f is definable over A in $\mathcal{M}$ iff its graph Γ_f is type-definable over A in $\mathcal{M}$.*

Definability of functions behaves well under extensions or restrictions:

Proposition 2. *if $f : M^n \to M$ is definable over A, then:*

(a) For any $\mathcal{N} \preccurlyeq \mathcal{M}$ with $A \subseteq N$, the restriction of f to N is a function (its image stays in N) definable in $\mathcal{N}$ over A;

(b) For any $\mathcal{N} \preccurlyeq \mathcal{M}$, there is a function g extending f to N definable in $\mathcal{N}$ over A.

Combining those results, we see that the composition of two functions definable over A is definable over A:

- If f and g are definable over A, then their graphs are definable over A, and then $\Gamma_{f \circ g}$ is type definable:

$$\Gamma_{f \circ g} = \{(x, z) | \inf_y d(z, f(y)) < \epsilon \wedge d(y, g(x)) < \epsilon\}_{\epsilon > 0}$$

- Moving up to a sufficiently saturated extension $\mathcal{N} \succcurlyeq \mathcal{M}$, we have functions $\hat{f}$ and $\hat{g}$ extending f and g to $\mathcal{N}$; since the graph of their composition is type-definable over A, their composition is definable over A.

- Going down again to $\mathcal{M}$, the restriction of $\hat{f} \circ \hat{g}$ is definable over A.

Extension by definition

Moving from definability in a model, we consider definability in a theory: in all the following, $\mathcal{L}_0 \subseteq \mathcal{L}$ are signatures and T, T_0 are an $\mathcal{L}$-theory and an $\mathcal{L}_0$-theory.

Definition 3. We say that T_0 is the *restriction of T to $\mathcal{L}_0$*, or equivalently that T is a *conservative extension of* T_0, if for every closed $\mathcal{L}_0$-condition E we have:

$$T \vDash E \Leftrightarrow T_0 \vDash E.$$

For T to be a conservative extension of T_0 it suffices to have $T_0 \subseteq T$ and to have an extension of every model of T_0 to a model of T; it is however not necessary.

Definition 4. We say that an $\mathcal{L}$-formula $\varphi(x)$ is *definable in T over $\mathcal{L}_0$* if for each $\epsilon > 0$ there is an $\mathcal{L}_0$-formula $\psi(x)$ such that:

$$T \vDash (\sup_x |\varphi(x) - \psi(x)|) \leqslant \epsilon.$$

When φ is $P(x)$ for a predicate $P \in \mathcal{L}$ or $d(f(x), y)$ for a function $f \in \mathcal{L}$, we say that P or f are definable in T over $\mathcal{L}_0$.

Remark 5. Saying that predicates P and functions f are definable in $\mathrm{Th}(\mathcal{M})$ (seen as an $\mathcal{L}(M) \cup \{P, f\}$-theory) over $\mathcal{L}(A)$ is the same as saying they are definable in $\mathcal{M}$ over A.

Definition 6. We say that T is an *extension by definitions* of T_0 if T is a conservative extension of T_0 such that every nonlogical symbol of $\mathcal{L}$ is defined in T over $\mathcal{L}_0$.

If T is a conservative extension of T_0, consider $\pi_n : S_n(T) \to S_n(T_0)$ defined by $\pi_n(p) = \{\varphi \in p \mid \varphi \text{ an } \mathcal{L}_0\text{-formula}\}$:

- π_n is continuous with respect to the logic topology: every closed set in $S_n(T_0)$ is $C_\Gamma(T_0) = \{p \in S_n(T_0) \mid \Gamma \subseteq p\}$ for some $\mathcal{L}_0$-partial type Γ, now the preimage of such a set is $C_\Gamma(T)$.

- π_n is surjective: $\pi_n(S_n(T)) = C_\Gamma(T_0)$ for some $\mathcal{L}_0$-partial type Γ since it is a closed subset of $S_n(T_0)$. Now for any $\varphi(x) \in \Gamma$, φ is in every type of $S_n(T)$, so we have $T \vDash \sup_x \varphi(x) = 0$, therefore the same holds for T_0 and any $p \in S_n(T_0)$ contains Γ.

Proposition 7. *If T is an extension by definitions of T_0, then π_n is injective and every $\mathcal{L}$-formula is defined in T over $\mathcal{L}_0$.*

Proof. Take p_1, $p_2 \in S_n(T)$ with $\pi_n(p_1) = \pi_n(p_2)$, take realisations a_1 and a_2 in models $\mathcal{M}_1$, $\mathcal{M}_2 \vDash T$.

- $(\mathcal{M}_1|_{\mathcal{L}_0}, a_1) \equiv (\mathcal{M}_2|_{\mathcal{L}_0}, a_2)$ since any $\mathcal{L}_0(a_1)$-formula verified by $\mathcal{M}_1$ is in the type of a_1 which is the same as the type of a_2;

- There exists an isomorphism $f : (\mathcal{M}_1|_{\mathcal{L}_0}, a_1)_D \to (\mathcal{M}_2|_{\mathcal{L}_0}, a_2)_D$ for some ultrafilter D;

- f extends uniquely to $(\mathcal{M}_1)_D$ since T is an extension by definitions of T_0, now let $\mathcal{N} = f((\mathcal{M}_1)_D)$;

- Now $\mathcal{N}$ is a model of T and $\mathcal{N}|_{\mathcal{L}_0} = (\mathcal{M}_2|_{\mathcal{L}_0})_D = (\mathcal{M}_2)_D|_{\mathcal{L}_0}$, and since any model of T is uniquely determined by its reduced to $\mathcal{L}_0$, we have $\mathcal{N} = (\mathcal{M}_2)_D$;

- f is an isomorphism form $(\mathcal{M}_1)_D$ to $(\mathcal{M}_2)_D$ mapping a_1 to a_2, so they must have the same type in $S_n(T)$, so to say $p_1 = p_2$.

Let φ be an $\mathcal{L}$-formula, and consider $\Phi = \widetilde{\varphi} \circ \pi_n^{-1}$, which exists since π_n is bijective. $\Phi : S_n(T_0) \to [0,1]$ is continuous, therefore there exists some $\mathcal{L}_0$-formulas $(\psi_k)_{k\in\mathbb{N}}$ such that $\widetilde{\psi_k} \to \Phi$ uniformely. Now:

$$\sup_{p\in S_n(T_0)} \left|\widetilde{\psi}(p) - \Phi(p)\right| < \epsilon \Rightarrow \sup_{p\in S_n(T)} \left|\widetilde{\psi}(\pi_n(p)) - \Phi(\pi_n(p))\right| < \epsilon$$

But $\Phi \circ \pi_n$ is just $\widetilde{\varphi}$ and $\widetilde{\psi} \circ \pi_n$ is just $\widetilde{\psi}$ when ψ is seen as an $\mathcal{L}$-formula. □

Corollary 8. *Let $(T_\alpha)_{\alpha<\gamma}$ be theories such that $T_{\alpha+1}$ is an extension by definition of T_α and $T_\lambda = \bigcup_{\alpha<\lambda} T_\alpha$ for λ a limit ordinal. Then any T_α is an extension by definition of T_0.*

Proof. By induction:

- If T_α is an extension by definitions of T_0 and $T_{\alpha+1}$ is an extension by defintion of T_α, every nonlogical symbol of $\mathcal{L}_{\alpha+1}$ is approached (for $\frac{\epsilon}{2}$) by an $\mathcal{L}_\alpha$-formula, which is then approached by an $\mathcal{L}_0$-formula.

- If T_α is an extension by definitions of T_0 for all $\alpha < \lambda$ limit, any nonlogical symbol in $\mathcal{L}_\lambda$ lies in some $\mathcal{L}_\alpha$.

□

Corollary 9. *If T is an extension by definitions of T_0, then every model of T_0 can be extended uniquely to a model of T.*

Proof. Unicity was done before. Existence: let $\mathcal{M}_0 \vDash T_0$. As seen before π_0 is bijective, so $T_1 = \pi_0^{-1}(\mathrm{Th}(\mathcal{M}_0)) \in S_n(T)$ is the unique completion of $T \cup \mathrm{Th}(\mathcal{M}_0)$. Let $\mathcal{M} \vDash T_1$ be $|M_0|^+$-saturated, and assume $\mathcal{M}_0 \preccurlyeq \mathcal{M}|_{\mathcal{L}_0}$ (in general they are elementarily equivalent). Now by a stronger version of proposition 2, M_0 is closed under $f^{\mathcal{M}}$ for function symbols $f \in \mathcal{L}$. Hence we can interpret symbols of $\mathcal{L}$ in $\mathcal{M}_0$ by reducting them from $\mathcal{M}$. Now by proposition 7 every $\mathcal{L}$-formula is defined in T over $\mathcal{L}_0$, therefore $\mathcal{M}_0$ extended this way is a model of T. □

Remark 10. A standard way to create extensions by defintions would be to add definable predicates or functions to the language; one can show that it works when adding to the theory axioms saying that our new symbol is approached by the sequence defining it.

Note also that the converse of corollary 9 hold, one can show it with a continuous version of Beth's theorem.

Algebraic and definable closures

Definition 11. We say that $a \in M^n$ is *definable* in $\mathcal{M}$ over A if $\{a\}$ is definable in $\mathcal{M}$ over A, and *algebraic* in $\mathcal{M}$ over A if it is contained in a compact set $C \subseteq M^n$ definable in $\mathcal{M}$ over A.

Equivalently we could define it on the coordinates:

Proposition 12. *$a = (a_1, \cdots, a_n)$ is A-definable/algebraic iff every a_i is A-definable/algebraic.*

Proof. If a is A-definable, then $d(x_i, a_i) = \inf_{y_1} \cdots \inf_{y_n} d((y, \cdots, x_i, \cdots, y_n), a)$ is definable; conversely if each a_i is A-definable, then $d(x, a) = \max_i(d(x_i, a_i))$ is A-definable.

If $C \subseteq M^n$ is a compact A-definable set, we want to prove that each projection C_i is compact (clear) and A-definable. The following is a definable predicate:

$$P_i(x) = \inf_{y \in C} d(x_i, y_i)$$

Now $\operatorname{dist}(x_i, C_i) = P_i(x_i, \cdots, x_i)$ and therefore a A-algebraic implies each a_i A-algebraic.

If $C_i \subseteq M$ are compact A-definable sets, then $C = C_1 \times \cdots \times C_n$ is compact, and:

$$\operatorname{dist}(x, C) = \inf_{y \in C} \max_i (d(x_i, y_i))$$

Therefore C is A-definable and if every a_i is A-algebraic, then a as well. $\square$

Remark 13. Instead of working with definable sets, we could use zerosets; in sufficiently saturated (namely, $\aleph_1$) models, these notions agree.

We write $\operatorname{dcl}_{\mathcal{M}}(A)$ for the set of all A-definable elements of M and $\operatorname{acl}_{\mathcal{M}}(A)$ for the set of all A-definable elements of M. We will see that these definable and algebraic closures don't depend on $\mathcal{M}$, and will therefore drop the subscript in due time.

Proposition 14. *For any $\mathcal{N} \succcurlyeq \mathcal{M}$, if $C \subseteq N^n$ is A-definable in $\mathcal{N}$ and $C \cap M^n$ is compact, then already $C \subseteq M^n$.*

Proof. Let $Q : N^n \to [0, 1]$ be the A-definable predicate such that $Q(x) = \operatorname{dist}(x, C)$. Let P be its restriction to M^n, then $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$ and $P(x) = \operatorname{dist}(x, C \cap M^n)$. For $\epsilon > 0$, we can cover C by a finite number of open balls of centers $c_1, \cdots, c_m$ and radius ϵ. Now if $P(x) < \epsilon$, for some j we have $d(x, c_j) < \epsilon$; therefore:

$$(\mathcal{M}, P) \vDash \sup_x \min(\epsilon \dot{-} P(x), \min_j (d(x, c_j)) \dot{-} 2\epsilon) = 0$$

Now $(\mathcal{N}, Q)$ satisfies the same condition with Q, and open balls of centers $c_1, \cdots, c_m$ and radius 2ϵ cover C. This give us that every element of C is the limit of a sequence of elements of M^n, which in turn means $C \subseteq M^n$. $\square$

Corollary 15. *If $\mathcal{N} \succcurlyeq \mathcal{M}$, then:*

$$\mathrm{dcl}_{\mathcal{M}}(A) = \mathrm{dcl}_{\mathcal{N}}(A) \quad \& \quad \mathrm{acl}_{\mathcal{M}}(A) = \mathrm{acl}_{\mathcal{N}}(A)$$

Proof. By proposition 14, if C is compact and A-definable in $\mathcal{N}$, then $C \cap M^n$ is compact, so $C \subseteq M^n$; it is then A-definable in $\mathcal{M}$, and $\mathrm{acl}_{\mathcal{N}}(A) \subseteq \mathrm{acl}_{\mathcal{M}}(A)$. Since singletons are compact, also $\mathrm{dcl}_{\mathcal{N}}(A) \subseteq \mathrm{dcl}_{\mathcal{M}}(A)$.

For the other inclusion, let $C \subseteq M^n$ be compact and A-definable in $\mathcal{M}$. Let $P(x) = \mathrm{dist}(x, C)$, then $P : M^n \to [0, 1]$ is A-definable in $\mathcal{M}$. We can extend it to $Q : N^n \to [0, 1]$ A-definable in $\mathcal{N}$ with $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$. Let D be the zeroset of Q, we then have $Q(x) = \mathrm{dist}(x, D)$. But now by proposition 14 $D = D \cap M^n = C$, therefore C is indeed A-definable in $\mathcal{N}$. $\square$

Proposition 16. *(a) $A \subseteq \mathrm{dcl}(A)$;*

(b) if $A \subseteq \mathrm{dcl}(B)$ then $\mathrm{dcl}(A) \subseteq \mathrm{dcl}(B)$ (so $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$);

(c) if $a \in \mathrm{dcl}(A)$ then there is a countable $A_0 \subseteq A$ with $a \in \mathrm{dcl}(A_0)$;

(d) if A is a dense subset of B then $\mathrm{acl}(A) = \mathrm{dcl}(B)$.
The same is true when replacing dcl *by* acl.

Proposition 17. *Any elementary map $\alpha : A \to B$ extends to an elementary map $\alpha' : \mathrm{acl}(A) \to \mathrm{acl}(B)$. If α happens to be surjective, then α' also.*