Notes on definable valuations and NIPity in extensions of $\mathbb{Q}_p$

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2019

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1 Introduction

Some very important notions of Model Theory are expressed via combinatorics properties that a theory has, or more often, does not have. Examples include the order property and stability, the independence property and NIP-ity, the tree property and NTP2, among others. The very large framework of this notes is the open question "When is a pure field NIP?", and somewhat surprisingly henselian valuations arise from pure NIP fields. As such $\mathbb{Q}_p$ is the best example, since it is NIP as a pure field, and admits a definable henselian valuation.

We will see that many fields admit definable henselian valuations. Our goal is to define one in every algebraic extension of $\mathbb{Q}_p$ (except $\mathbb{Q}_p^{\mathrm{alg}}$); and in every NIP field modulo Shelah's conjecture (see conjecture 7.1). This is achievable thanks to a theorem of Jahnke and Koenigsmann in [8], which states that the so-called canonical p-henselian valuation is ring-definable on fields satisfying certain conditions, namely, writing T_p for the ring-theory of fields of characteristic p or containing a primitive p^{th}-root of unity, in case $p \neq 2$, the theorem applies to every $K \vDash T_p$. In case $p = 2$, we run into a problem if the residue field is Euclidian; we will see why and how to define in this case a slightly different valuation.

This being done, we will know that in any $\mathbb{Q}_p \subseteq K \subsetneq \mathbb{Q}_p^{\mathrm{alg}}$, K is NIP iff (K, v_p) is NIP. This is very useful since NIP henselian valued field are very well understood, and with the help of a very recent theorem by Franziska Jahnke and Sylvy Anscombe in [1], we will be able to pinpoint which extension of $\mathbb{Q}_p$ are NIP.

2 Canonical valuations

This section is mainly a rewriting of [5, sec. 4].

2.1 Henselian valuations

Definition 2.1.1. A valuation v on a field K is called *henselian* if it extends uniquely to the algebraic closure K^{alg}. We also say that a field is henselian if it admits a non-trivial henselian valuation.

Henselianity can be expressed via several equivalent properties:

Proposition 2.1.2 (Hensel's lemma)**.** *For a valued field (K, v), the following are equivalent:*

1. *v is henselian;*

2. *For all $P \in \mathcal{O}_v[X]$, if $\overline{P}$ has a simple zero $\alpha \in Kv$, then P has a zero $a \in \mathcal{O}_v$ with $\overline{a} = \alpha$;*

3. *For all $P \in \mathcal{O}_v[X]$, if there is $a \in \mathcal{O}_v$ such that $v(P(a)) > 2v(P'(a))$, then there is a unique $b \in \mathcal{O}_v$ with $P(b) = 0$ and $v(b - a) > v(P'(a))$;*

4. *If K^h is a henselization[1] of K, then $K = K^h$.*

Let us now state properties obtained by studying valuations:

Proposition 2.1.3. *Let (K, v) be a valued field.*

1. *If $K = K^{\mathrm{sep}}$, then $Kv = Kv^{\mathrm{alg}}$;*

2. *If v is the composition of two valuations v_1 and v_2, then v is henselian iff v_1 and v_2 are henselian.*

For more details and proofs of the previous results, see [5, sec. 4.2].

Since any two valuation rings are included in a bigger valuation ring, namely their product, and coarsenings of a valuation are linealry ordered, valuations are always arranged in a tree. Henselian valuation rings are well behaved regarding this tree structure, forming two meaningfull components. There, in the middle, lies one ring; one ring to compare them all and in the darkness define them.

All of the results of this subsection are based on [5, sec. 4.4].

Proposition 2.1.4. *If a field K admits two independent henselian valuation rings, then K is separably closed.*

Note that by proposition 2.1.3 any coarsening of a henselian valuation is still henselian. Now split the set H of all henselian valuation rings of K in two:

$$H_1 = \{\mathcal{O}_v \mid v \text{ is henselian and } Kv \neq Kv^{\mathrm{sep}}\}$$
$$H_2 = \{\mathcal{O}_v \mid v \text{ is henselian and } Kv = Kv^{\mathrm{sep}}\}$$

Since K itself is a valuation ring H is never empty.

Proposition 2.1.5. *Let K be a field, then H_1 is linearly ordered by inclusion; furthermore, for any $\mathcal{O}_1 \in H_1$ and $\mathcal{O}_2 \in H_2$, we have $\mathcal{O}_2 \subset \mathcal{O}_1$.*

It is now clear what we ought to define as the canonical henselian valuation: the finest valuation which does not branch, see fig. 1.

[1]See section 6.2 for a definition of the henselization.

Definition 2.1.6. The canonical henselian valuation of a field K, denoted by v_K, is the coarsest valuation of H_2 if H_2 is non-empty, and the finest valuation of H_1 if H_2 is empty.

Note that this valuation exists by Zorn's lemma in the first case and is just the intersection of all valuations of H_1 in the second case.

Proposition 2.1.7. *It follows from the definition:*

1. *Every henselian valuation is comparable with v_K and with every coarsening of it;*

2. *v_K is non-trivial iff $K \neq K^{\text{sep}}$ and K is henselian;*

3. *No proper coarsening of v_K has separably closed residue field;*

4. *All proper refinements of v_K have separably closed residue field.*

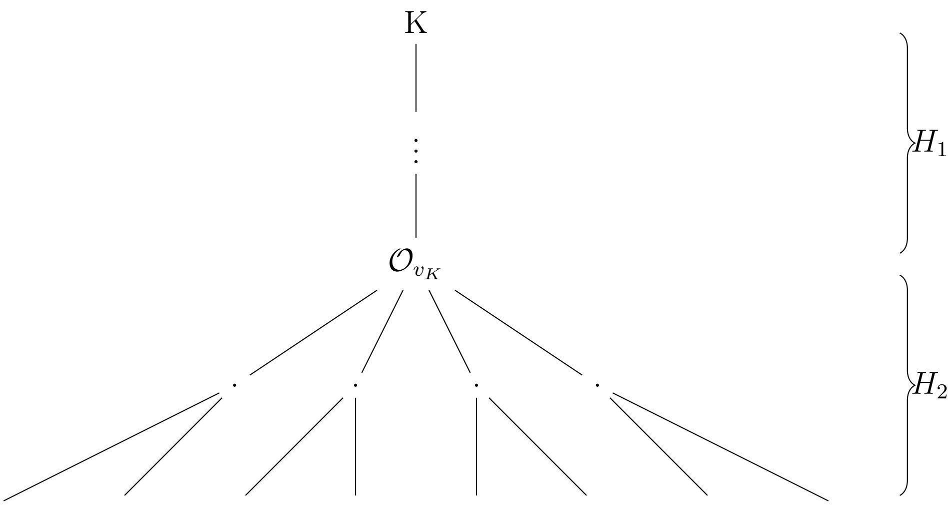

Figure 1: The tree structure of H.

2.2 p-henselian valuations

Definition 2.2.1. A valuation v on a field K is called *p-henselian* if it extends uniquely to the p-closure $K(p)$, which is the compositum of all Galois p-power degree extension of K. We also say that a field is p-henselian if it admits a non-trivial p-henselian valuation.

Similar to the henselian case, we have a number of properties equivalent to p-henselanity.

Proposition 2.2.2 (p-Hensel's lemma)**.** *For a valued field (K, v), the following are equivalent:*

1. *v is p-henselian;*

2. *For all $P \in \mathcal{O}_v[X]$ splitting in $K(p)$, if $\overline{P}$ has a simple zero $\alpha \in Kv$, then P has a zero $a \in \mathcal{O}_v$ with $\overline{a} = \alpha$;*

3. *For all $P \in \mathcal{O}_v[X]$ splitting in $K(p)$, if there is $a \in \mathcal{O}_v$ such that $v(P(a)) > 2v(P'(a))$, then there is a unique $b \in \mathcal{O}_v$ with $P(b) = 0$ and $v(b - a) > v(P'(a))$;*

4. *If K^h is a henselisation of K, then $K = K^h \cap K(p)$.*

Let us now state properties derived from the study of $K(p)$:

Proposition 2.2.3. *Let (K, v) be a valued field.*

1. *v is p-henselian iff it extends uniquely to every Galois extension of degree p;*

2. *If $K(p) = K$, then $Kv(p) = Kv$;*

3. *If v is the composition of two valuations v_1 and v_2, then v is p-henselian iff v_1 and v_2 are p-henselian.*

Now we can apply the same reasoning than for the henselian case to obtain a canonical valuation.

Proposition 2.2.4. *If a field K admits two independent p-henselian valuation rings, then K is p-closed.*

Note that by proposition 2.2.3 any coarsening of a p-henselian valuation is still p-henselian. Now split the set H^p of all p-henselian valuation rings of K in two:

$$H_1^p = \{\mathcal{O}_v \mid v \text{ is } p\text{-henselian and } Kv \neq Kv(p)\}$$
$$H_2^p = \{\mathcal{O}_v \mid v \text{ is } p\text{-henselian and } Kv = Kv(p)\}$$

Since K itself is a valuation ring H^p is never empty.

Proposition 2.2.5. *Let K be a field, then H_1^p is linearly ordered by inclusion; furthermore, for any $\mathcal{O}_1 \in H_1^p$ and $\mathcal{O}_2 \in H_2^p$, we have $\mathcal{O}_2 \subset \mathcal{O}_1$.*

We can now draw the same picture that in the henselian case, see fig. 1. Note that henselian valuations are in peculiar p-henselian.

Definition 2.2.6. The canonical p-henselian valuation of a field K, denoted by v_K^p, is the coarsest valuation of H_2^p if H_2^p is non-empty, and the finest valuation of H_1^p if H_2^p is empty.

Proposition 2.2.7. *It follows from the definition:*

1. *Every p-henselian valuation is comparable with v_K^p and with every coarsening of it;*

2. *v_K^p is non-trivial iff $K \neq K(p)$ and K is p-henselian;*

3. *No proper coarsening of v_K^p has p-closed residue field;*

4. *All proper refinements of v_K^p have p-closed residue field.*

3 Expressing p-henselianity in first-order

The first step in our quest to ring-define the canonical p-henselian valuation is to show how p-henselianity can be described by first-order valued-fields-formulas.

Most of the results of this section were obtained by Koenigsmann in [10].

3.1 p-henselianity of a valuation

Recall that by proposition 2.2.3 item 1 we only care about Galois extension of degree p. In general, those extensions can be quite wild; but when the field is of characteristic p then they are exactly of the form $K(\alpha)$, where α is a root of an Artin-Schreier Polynomial $X^p - X - a$. Similarily, if K contains a primitive p^{th}-root of unity – which we will denote by ζ_p from now on – then all Galois extensions of degree p are of the form $K(\alpha)$ for a root of $X^p - a$. This leads to a first description of p-henselianity:

Lemma 3.1.1. *Let (K, v) be a valued field such that $\mathrm{ch}(Kv) \neq p$ and $\zeta_p \in K$, then:*

$$v \text{ is } p\text{-henselian} \Leftrightarrow 1 + \mathcal{M}_v \subseteq (K^\times)^p.$$

Proof. If v is p-henselian, take $m \in \mathcal{M}_v$ and consider $X^p - (1 + m)$; it has a root by proposition 2.2.2 item 2, so $1 + \mathcal{M}_v \subseteq (K^\times)^p$.

Conversely, suppose $1 + \mathcal{M}_v \subseteq (K^\times)^p$, let L/K be a Galois extension of degree p such that $L \subseteq K^h$, and take $w = v^h|_L$. Since $\zeta_p \in K$, $L = K(\sqrt[p]{a})$

for some $a \in K \setminus (K^\times)^p$. Now since K^h is an immediate extension, $L \subseteq K^h$ is also immediate, so $wL = vK$ and $w(\sqrt[p]{a}) = v(b)$ for some $b \in K$; we may therefore replace a by $b^{-p}a$ in order to assume $a \in \mathcal{O}_v^\times$. On the other hand $Kv = Lw$, thus $\overline{\sqrt[p]{a}} = \overline{c}$ for some $c \in K$; now we may replace a by $c^{-p}a$ and assume $a \in 1 + \mathcal{M}_v \subseteq (K^\times)^p$, which contradicts $[L : K] = p$. Therefore there is no extension of degree p inside K^h, which means $K(p) \cap K^h = K$; therefore v is p-henselian. □

Lemma 3.1.2. *Let (K, v) be a valued field such that $\mathrm{ch}(K) = p$, then:*

$$v \textit{ is } p\textit{-henselian} \Leftrightarrow \mathcal{M}_v \subseteq K^{(p)} = \{x^p - x \mid x \in K\}.$$

Proof. If v is p-henselian, take $m \in \mathcal{M}_v$ and consider $X^p - X - m$; it has a root by proposition 2.2.2 item 2, so $\mathcal{M}_v \subseteq K^{(p)}$.

The proof of the converse direction is due to Chatzidakis and Perara in [3]: suppose $\mathcal{M}_v \subseteq K^{(p)}$, let L/K be an immediate Galois extension of degree p, and take w any extension of v to L. Since $\mathrm{ch}(K) = p$, $L = K(\alpha)$ with $\alpha^p - \alpha = a \notin K^{(p)}$.

Step 1: we may assume L/K immediate. The fundamental equality gives:

$$p = [L : K] = [wL : wK][Lw : Kv]p^d n$$

where n is the number of extensions of v to L. If L/K is not immediate, then either $[wL : wK]$ or $[Lw : Kv]$ is bigger than 1, hence equal to p, thus $n = 1$ and we have p-henselianity.

Step 2: the set $C = v(K^{(p)} - a)$ admits 0 as a (strict) upper bound but has no max element. Suppose $v(x^p - x - a) > 0$ for some $x \in K$. Then, since $\mathcal{M}_v \subseteq K^{(p)}$, we have $x^p - x - a = y^p - y$ for some $y \in K$, and thus $a = (x - y)^p - (x - y) \in K^{(p)}$, which contradicts our choice of α. So 0 is a (large) upper bound of C.

Now let $b \in K$ and $\gamma = v(b^p - b - a)$. We have $w(b - \alpha) \in wL = vK$, so there exists $x \in K$ such that $w(b - \alpha) = v(x)$. Now $w(\frac{b-\alpha}{x}) = 0$, thus $\overline{\frac{b-\alpha}{x}} \in Lw = Kv$ and there exists $y \in K$ such that $\overline{\frac{b-\alpha}{x}} = \overline{y}$. This yields $w(\frac{b-\alpha}{x} - y) > 0$, or $w(b - xy - \alpha) > v(x) = w(b - \alpha)$. Write $c = -xy$.

We claim that $w(b-\alpha) < 0$. Indeed, if $w(b-\alpha) \geqslant 0$, then $w(b+c-\alpha) > 0$. But $w((b+c-\alpha)^p-(b+c-\alpha)) = v((b+c)^p-(b+c)-a) \in C$ can't be positive as shown before. Note: since there is nothing special about b, the same argument would work for any $z \in K$, in peculiar for $b + c$: $w(b + c - \alpha) < 0$.

Now $\gamma = v(b^p - b - a) = w((b - \alpha)^p - (b - \alpha)) = pw(b - \alpha)$, and $v((b + c)^p - (b + c) - a) = w((b + c - \alpha)^p - (b + c - \alpha)) = pw(b + c - \alpha) >$

$pw(b-\alpha)=\gamma$. Thus, C can't have a max element; in peculiar 0 is a strict upper bound.

Step 3: we define a "good" sequence in K. Our purpose is to apply the following, which is a reformulation of [9, thm. 3]:

Fact. Let $(x_\lambda)_{\lambda<\kappa}$ be a pseudo-Cauchy sequence without pseudo-limit in K such that $(v(f(x_\lambda)))_{\lambda<\kappa}$ is stritcly increasing for some $f \in K[X]$. Let $P(X) \in K[X]$ non-constant be of minimal degree such that $(P(x_\lambda))_{\lambda<\kappa}$ admits 0 as a pseudo-limit. Then there exist an immediate extension $(K(x_\infty), \tilde{v})$ of (K, v), which checks and is uniquely determined by the conditions $P(x_\infty) = 0$ and x_∞ is a pseudo-limit of $(x_\lambda)_{\lambda<\kappa}$.

So we aim to find such a sequence, with α as a pseudo-limit. Let $(c_\lambda)_{\lambda<\kappa}$ incresingly enumerate C, and choose x_λ for each $\lambda < \kappa$ such that $c_\lambda = v(x_\lambda^p - x_\lambda - a)$. By what was done before, we know $c_\lambda = v(x_\lambda^p - x_\lambda - a) = pw(x_\lambda - \alpha)$. For all $\lambda < \mu < \kappa$, we know $c_\lambda < c_\mu$, hence:

$$v(x_\lambda - x_\mu) = w(x_\lambda - \alpha - x_\mu + \alpha) = \frac{1}{p}c_\lambda$$

So $(x_\lambda)_{\lambda<\kappa}$ is pseudo-Cauchy, and we note $\gamma_\lambda = v(x_\lambda - x_\mu) = \frac{1}{p}c_\lambda$. Furthermore, if $f(X) = X^p - X - a$, we have $v(f(x_\lambda)) = c_\lambda$ strictly increasing.

Finally, if $l \in K$ is a pseudo-limit of $(x_\lambda)_{\lambda<\kappa}$, then $v(l^p - l - a) \geqslant c_\lambda$ for all $\lambda < \kappa$. But then it is a max element of C, which can't be. So $(x_\lambda)_{\lambda<\kappa}$ does not have a pseudo-limit in K, and thus we can apply the previous fact to it. We would like to apply it while taking $P(X) = X^p - X - a$. For that, we need to show that $P(x_\lambda)_{\lambda<\kappa}$ admits 0 as pseudo-limit, and that no polynomial of smaller degree does.

Step 4: if $Q \in K_{p-1}[X]$, then $v(Q(x_\lambda))_{\lambda<\kappa}$ is ultimately constant. Clearly, this is true for polynomials of degree 0 or 1. Let $1 < n < p$, suppose it is true for all polynomial of degree smaller than n, and take Q of degree n. Suppose $v(Q(x_\lambda))_{\lambda<\kappa}$ is not ultimately constant. For $\lambda < \kappa$, we then have utlimately:

$$v(Q(x_\lambda)) = v(Q'(\lambda)) + \gamma_\lambda = \delta' + \gamma_\lambda$$

This is a consequence of [9, lem. 8], and recall that Q' is of degree $< n$ so $\delta' = v(Q'(\lambda))$ does not depend on λ. We then write:

$$P(X) = X^p - X - a = \sum_{i=1}^{p-n} R_i(X)Q(X)^i$$

with $R_i \in K_{n-1}[X]$. Thus, ultimately $v(R_i(x_\lambda)) = \delta_i$ is constant, and ultimately $v(R_i(x_\lambda)Q(x_\lambda)^i) = \delta_i + i(\delta' + \gamma_\lambda)$. Thus, utlimately:

$$v(P(x_\lambda)) = v(\sum_{i=1}^{p-n} R_i(x_\lambda)Q(x_\lambda)^i) = \delta_{i_0} + i_0(\delta' + \gamma_\lambda)$$

For some $1 \leqslant i_0 \leqslant p-n$ (see [9, lem. 4]). But $v(P(x_\lambda)) = c_\lambda = p\gamma_\lambda$, so ultimately $(p-i_0)\gamma_\lambda = \delta_{i_0}$, which is impossible since γ_λ is strictly increasing.

Hence, if $Q \in K_{p-1}[X]$, then $v(Q(x_\lambda))_{\lambda<\kappa}$ is ultimately constant, so it can't have 0 as a pseudo-limit. On the other hand, $v(P(x_\lambda))_{\lambda<\kappa}$ is strictly increasing, thus admits 0 as a pseudo-limit; we can then apply the fact to $(x_\lambda)_{\lambda<\kappa}$ with this P. It is clear that $x_\infty = \alpha$ will work. We thus get an immediate extension $(K(a), \tilde{v})$ of (K, v). Since (L, w) checks the conditions uniquely determining $(K(a), \tilde{v})$, they must be the same. Now any other extension w' of v to L also check those properties, hence $w' = w$ and v is p-henselian. □

Lemma 3.1.3. *Let (K, v) be a valued field such that $\mathrm{ch}(K) = 0$, $\mathrm{ch}(Kv) = p$, $\zeta_p \in K$ and v is of rank 1, then:*

$$v \textit{ is } p\textit{-henselian} \Leftrightarrow 1 + p^2\mathcal{M}_v \subseteq (K^\times)^p.$$

Proof. If v is p-henselian, take $m \in \mathcal{M}_v$ and consider $f = X^p - (1+p^2m)$; now $v(f(1)) = v(-p^2m) > 2v(p) = 2v(f'(1))$, so it has a root by proposition 2.2.2 item 3. Note that this also works when v is not of rank 1.

Conversely, suppose $1 + p^2\mathcal{M}_v \subseteq (K^\times)^p$, and take $L = K(\sqrt[p]{a})$ as before; we may assume $a \in 1 + \mathcal{M}_v$. Now consider the Cauchy completion $(\hat{K}, \hat{v})$ of (K, v) which exists since v is of rank 1. The completion is always henselian, thus (K^h, v^h) embeds uniquely in $(\hat{K}, \hat{v})$; we may therefore assume $L \subseteq \hat{K}$.

By density of K in $\hat{K}$, we can take $b \in K$ such that $\hat{v}(b - \sqrt[p]{a}) > v(p^2)$, so to say $b \in \sqrt[p]{a} + p^2\mathcal{M}_{\hat{v}}$. Then $b^p \in a + p^2\mathcal{M}_{\hat{v}}$, and since b and a are in K, $b^p \in a + p^2\mathcal{M}_v = a(1 + p^2\mathcal{M}_v) \subseteq a(K^\times)^p$. This means $a \in (K^\times)^p$, and therefore $L = K$ and v is p-henselian. □

We will then combine the three cases in order to have a criterion for any (K, v) of characteristic p or containing ζ_p. The most troublesome case will be when (K, v) is of mixed characteristic $(0, p)$ with valuation of rank bigger than 1; in which case we define:

$$\Delta_1 = \bigcup_{\substack{\Delta \leqslant vK \text{ cvx} \\ v(p) \notin \Delta}} \Delta, \text{ and } \Delta_2 = \bigcap_{\substack{\Delta \leqslant vK \text{ cvx} \\ v(p) \in \Delta}} \Delta.$$

We denote the associated valuations by v_1 and v_2, and the residue fields K_1 and K_2. We can now write v as a composition of valuations, in order to have (K, v_2) of equicharacteristic 0, $(K_1, \overline{v})$ of equicharacteristic p, and most importantly $(K_2, \overline{v_1})$ of mixed characteristic and of rank 1.

$$\begin{tikzcd}
K \arrow[r, "v"] \arrow[d, "v_2"] & Kv \\
K_2 \arrow[r, "\overline{v_1}"] & K_1 \arrow[u, "\overline{v}"']
\end{tikzcd}$$

Proposition 3.1.4. *Let (K, v) be a valued field of characteristic p or containing ζ_p, then v is p-henselian iff $1+p^2\mathcal{M}_v \subseteq (K^\times)^p$ and $\mathcal{M}_v \subseteq K^{(p)}+p\mathcal{M}_v$.*

Proof. If (K, v) is not of mixed characteristic $(0, p)$, then it is an immediate consequence of the previous lemmas: when $\mathrm{ch}(Kv) \neq p$, $v(p) = 0$ so $p\mathcal{M}_v = \mathcal{M}_v$ and we conclude by lemma 3.1.1; and when $\mathrm{ch}(k) = p$, $p = 0$ so $p\mathcal{M}_v = \{0\}$ and we conclude by lemma 3.1.2.

If (K, v) is of mixed characteristic $(0, p)$, then we construct v_1 and v_2 as above (note that $\overline{v}$ and v_2 may be trivial if v is already of rank 1). Now, by composition, v is p-henselian iff $\overline{v}$, $\overline{v_1}$ and v_2 are p-henselian, and by the three previous lemmas:

$$v \text{ is } p\text{-henselian } \Leftrightarrow \left\{ \begin{array}{l} 1+\mathcal{M}_{v_2} \subseteq (K^\times)^p \\ 1+p^2\mathcal{M}_{\overline{v_1}} \subseteq (K_2^\times)^p \\ \mathcal{M}_{\overline{v}} \subseteq K_1^{(p)} \end{array} \right.$$

We know that if v is p-henselian, then $1+p^2\mathcal{M}_v \subseteq (K^\times)^p$ by the proof of lemma 3.1.3. Now, since $\mathcal{M}_{v_2} \subseteq \mathcal{M}_{v_1} \subseteq \mathcal{M}_v$ and $v_2(p) = 0$, we have that:

$$1+p^2\mathcal{M}_v \subseteq (K^\times)^p \Rightarrow \left\{ \begin{array}{l} 1+\mathcal{M}_{v_2} \subseteq (K^\times)^p \\ 1+p^2\mathcal{M}_{\overline{v_1}} \subseteq (K_2^\times)^p \end{array} \right.$$

Furthermore, lifting on one way and projecting to residues on the other, we see that:

$$\mathcal{M}_{\overline{v}} \subseteq K_1^{(p)} \Leftrightarrow \mathcal{M}_v \subseteq K^{(p)} + \mathcal{M}_{v_1}.$$

We thus have:

$$v \text{ is } p\text{-henselian } \Leftrightarrow \left\{ \begin{array}{l} 1+p^2\mathcal{M}_v \subseteq (K^\times)^p \\ \mathcal{M}_v \subseteq K^{(p)} + \mathcal{M}_{v_1} \end{array} \right.$$

We now use a completion method to establish that $\mathcal{M}_v \subseteq K^{(p)} + \mathcal{M}_{v_1} \Leftrightarrow \mathcal{M}_v \subseteq K^{(p)} + p\mathcal{M}_v$: suppose $\mathcal{M}_v \subseteq K^{(p)} + \mathcal{M}_{v_1}$ and take $a \in \mathcal{M}_v$. Let

$f = X^p - X - a \in K[X]$ and let f_1, f_2 be the residues of f in K_1 and K_2. Since $a = x^p - x + m$ for some $x \in K$ and $m \in \mathcal{M}_{v_1}$, we have that f_1 has a root, and since $(K_2, \overline{v_1})$ is of rank 1, f_2 will have a root α in the completion $(\hat{K_2}, \hat{\overline{v_1}})$. We can approximate α by some $b \in K_2$ such that $\hat{\overline{v_1}}(b - \alpha) > p$. Now $b = \alpha + pm'$ for some $m' \in \mathcal{M}_{\hat{\overline{v_1}}}$, therefore:

$$
\begin{aligned}
b^p - b &= (\alpha - pm')^p - (\alpha - pm') \\
&= (\alpha^p - p\alpha^{p-1}pm' + \cdots + (-pm')^p) - \alpha + pm' \\
&= \alpha^p - \alpha + p(-\alpha^{p-1}pm' + \cdots + (-m')^p p^{p-1} + m') \\
&= \overline{a} + pm''
\end{aligned}
$$

where $\hat{\overline{v_1}}(m'') > 0$, and since $b, \overline{a} \in K_2$, also $m'' \in K_2$. So $\overline{a} = b^p - b - pm'' \in K_2^{(p)} + p\mathcal{M}_{\overline{v_1}}$, and lifting it we have $a \in K^{(p)} + p\mathcal{M}_{v_1}$. Finally $p\mathcal{M}_{v_1} \subseteq p\mathcal{M}_v \subseteq \mathcal{M}_{v_1}$, and we conclude. □

3.2 p-henselianity of a field

Definition 3.2.1. The p-topology of a field K, denoted τ_p, is defined in the following way:

1. If $\zeta_p \in K$, τ_p is the coarsest topology for which $(K^\times)^p$ is open and all linear transformations are continuous; a subbase for τ_p is given by sets $a(K^\times)^p + b$ for $a \in (K^\times)^p$ and $b \in K$

2. If $\mathrm{ch}(K) = p$, τ_p is the coarsest topology for which $K^{(p)}$ is open and all Möbius transformations are continuous; a subbase for τ_p is given by sets $\left\{\frac{ax+b}{cx+d} \mid x \in K^{(p)},\ x \neq -\frac{d}{c}\right\}$ for $a, b, c, d \in K$ with $ad \neq bc$.

Theorem 3.2.2. *Let v be a non-trivial valuation on K inducing the topology τ_v, then $\tau_p = \tau_v$ iff some non-trivial coarsening w of v is p-henselian.*

In this case, τ_p admits a nice base: when $\mathrm{ch}(K) \neq p$, this base is formed by all the sets $(a(K^\times)^p + b) \cap (c(K^\times)^p + d)$ with $c, d \neq 0$; when $\mathrm{ch}(K) = p$ by all the sets $\left\{\frac{ax+b}{cx+d} \mid x \in K^{(p)},\ x \neq -\frac{d}{c}\right\}$ with $ad \neq bc$.

Proof. Suppose $\tau_v = \tau_p$. In the case $\mathrm{ch}(K) \neq p$, then $(K^\times)^p$ must be open for τ_v, so there is a τ_v-open neighbourhood of 1 included in $(K^\times)^p$; so there is an $\mathcal{O}_v$-ideal non-trivial $\mathcal{A}$ such that $1 + \mathcal{A} \subseteq (K^\times)^p$, let's suppose it maximal for this property. We start with a preliminary statement:

$$b^2 \in \mathcal{A} \Rightarrow pb \in \mathcal{A}$$

Since $\mathcal{A}$ is an $\mathcal{O}_v$-ideal, $a \in \mathcal{A}$ implies $a\mathcal{O}_v \subseteq \mathcal{A}$, in particular any $c \in K$ such that $v(c) \geqslant v(b)$ verify $c^2 \in \mathcal{A}$, and since $1 + (-1) = 0 \notin (K^\times)^p$, we know that $\mathcal{A} \subseteq \mathcal{M}_v$, so $v(c) \geqslant v(b) > 0$. Now:

$$(1+c)^p = 1 + pc + \binom{p}{2} c^2 + \cdots + c^p$$

Since $v(1+pc) = 0$, $(1+pc)\mathcal{A} = \mathcal{A}$. Now $\binom{p}{2}c^2 + \cdots + c^p \in \mathcal{A}$, so

$$(1+c)^p \in (1+pc) + \mathcal{A} = (1+pc)(1+\mathcal{A}) \subseteq (1+pc)(K^\times)^p$$

Therefore $1+pc \in (K^\times)^p$ for each $c \in b\mathcal{O}_v$, therefore $1 + pb\mathcal{O}_v \subseteq (K^\times)^p$, and by maximality $pb\mathcal{O}_v \subseteq \mathcal{A}$; this proves the statement.

If $p\mathcal{A} = \mathcal{A}$, then $\mathcal{A}$ is stable by square roots; so $\mathcal{A}$ is a radical ideal, therefore prime: if $ab \in \mathcal{A}$, suppose with $v(a) \geqslant v(b)$, then $ab^{-1} \in \mathcal{O}_v$ so $abab^{-1} = a^2 \in \mathcal{A}$, thus $a \in \mathcal{A}$. Take w the coarsening of v such that $\mathcal{M}_w = \mathcal{A}$, now $\mathrm{ch}(Kw) \neq p$: if $w(p) > 0$ then $\inf(w(\mathcal{A})) = \inf(w(p\mathcal{A})) > \inf(w(\mathcal{A}))$. This coarsening is p-henselian since $1 + \mathcal{M}_w \subseteq (K^\times)^p$ (see lemma 3.1.1).

If $p\mathcal{A} \subsetneq \mathcal{A}$, we must have $v(p) > 0$, so to say $\mathrm{ch}(Kv) = p$. Consider the coarsening w of v with $\mathcal{M}_w = \sqrt{p\mathcal{A}}$. Then $p^2\mathcal{M}_w \subseteq \mathcal{A}$. Indeed, take $m \in \mathcal{M}_w$, we have $m^2 \in p\mathcal{A} \subsetneq \mathcal{A}$, thus $pm \in \mathcal{A}$ and $p^2 m \in p\mathcal{A} \subsetneq \mathcal{A}$. Now using the same technique than in the proof of proposition 3.1.4, we can reduce to the case where w is of rank 1, and then w is p-henselian by lemma 3.1.3, and we are done with the case $\mathrm{ch}(K) \neq p$.

In the case $\mathrm{ch}(K) = p$, then $K^{(p)}$ must be open for τ_v, so there is a τ_v-open neighbourhood of 0 included in $K^{(p)}$; so there is an $\mathcal{O}_v$-ideal non-trivial $\mathcal{A}$ such that $\mathcal{A} \subseteq K^{(p)}$, let's suppose it maximal for this property.

Now if $b^p \in \mathcal{A}$ then any c with $v(c) \geqslant b$ checks $c = c^p - (c^p - c) \in \mathcal{A} + K^{(p)} = K^{(p)}$. So $b\mathcal{O}_v \in K^{(p)}$ and thus $b\mathcal{O}_v \in \mathcal{A}$ by maximality. So $\mathcal{A}$ is a radical ideal, hence prime, and taking $\mathcal{M}_w = \mathcal{A}$ yields w p-henselian.
For the converse, suppose w is a non-trivial p-henselian coarsening of v. Then $\tau_v = \tau_w$, so we may as well take $v = w$. Then in case $\mathrm{ch}(K) = p$, $\mathcal{M}_v \subseteq K^{(p)} = \bigcup_{x \in K^{(p)}} (x + \mathcal{M}_v)$, and in case $\mathrm{ch}(K) \neq p$, $1 + p^2\mathcal{M}_v \subseteq (K^\times)^p = \bigcup_{x \in (K^\times)^p} x(1 + p^2\mathcal{M}_v)$. So we have $\tau_p \subseteq \tau_v$.

To see that $\tau_p \subseteq \tau_v$, it suffices to check $\mathcal{M}_v \subseteq \tau_p$; and for this it suffices to find an open τ_p-neighbourhood of 0 $U \subseteq \mathcal{M}_v$, since then $\mathcal{M}_v = \bigcup_{x \in \mathcal{M}_v} (x + U)$.

In case $\mathrm{ch}(K) \neq p$, then we chose $a \in p^2\mathcal{M}_v \setminus K^p$, and $U = a(1 - (K^\times)^p) \cap a^2(1 - (K^\times)^p)$ works.

In case $\mathrm{ch}(K) = p$, then we chose $a \in K \setminus (\mathcal{M}_v \cup K^{(p)})$, and $U = \left\{ \frac{a^2 x}{x + a^{-1}} \mid x \in K^{(p)} \right\}$ works.

All the details and (long) calculations can be found in the original paper [10] by Koenigsmann. □

Corollary 3.2.3. *When $p \neq 2$, there is a first-order-ring-sentence expressing the fact that a field $K \neq K(p)$, $K \vDash T_p$ (see section 1) is p-henselian; namely, this sentence reads "τ_p is a V-topology".*

When $p = 2$, the sentence "τ_p is a V-topology" might not work when K is euclidean, but it still expresses p-henselianity for fields $K \neq K(2)$, $K \vDash T_2$ which are non-euclidean.

Proof. V-topologies are exactly the topologies induced by valuations or archimedean absolute values. Since we threw the euclidean case out of the window, no archimedean absolute value can exist, therefore τ_p is a V-topology iff $\tau_p = \tau_v$ for some valuation v, and by theorem 3.2.2 K is p-henselian.

We still have to check that this is a first-order-ring-sentence, but once again theorem 3.2.2 gives us a nice base for τ_p, and being a V-topology is expressible just in term of the base. All the claims above and more information on V-topologies can be found in [5, App. B] □

4 Ring-defining v_K^p

4.1 Overview of the proof

Following Jahnke and Koenigsman in [8], the final step in our quest will be to exhibit a valued-field-sentence characterising v_K^p, and apply afterwards Beth's definability theorem:

Theorem 4.1.1 (Beth)**.** *Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Let $\mathcal{L}_P = \mathcal{L} \cup \{P\}$, where P is a new unary predicate symbol, and let $T_P \supseteq T$ be an $\mathcal{L}_P$-theory.*

If every model $\mathcal{M}$ of T can be extended uniquely to a model $\mathcal{M}_P = (\mathcal{M}, P)$ of T_P, then P is already $\mathcal{L}$-definable modulo T: there is an $\mathcal{L}$-formula φ such that if $\mathcal{M} \vDash T$, then $\varphi(M) = P^{\mathcal{M}_P}$.

Taking $\mathcal{L}_{\text{ring}}$ for $\mathcal{L}$, T_p for T and adding a new predicate symbol $\mathcal{O}_v$, we want to axiomatise the property $\mathcal{O}_v = \mathcal{O}_{v_K^p}$; we claim that this is done in the case $p \neq 2$ by the following parameter-free sentence ψ_p:

1. If $K = K(p)$ then $\mathcal{O}_v = K$, and
2. if $K \neq K(p)$ then:
   (a) $\mathcal{O}_v$ is a valuation ring of K, and

(b) v is p-henselian, and

(c) if $Kv \neq Kv(p)$, then Kv is not p-henselian, and

(d) if $Kv = Kv(p)$, then:

i. vK has no non-trivial p-divisible convex subgroup, or

ii. it has one and:

A. $\mathrm{ch}(K) = p$ and $\forall x \in \mathcal{M}_v \setminus \{0\}$, $x^{-1}\mathcal{O}_v \nsubseteq K^{(p)}$, or

B. (K, v) is of mixed characteristic p and Kv is not perfect, or

C. (K, v) is of mixed characteristic p, Kv is perfect and $\forall x \in \mathcal{M}_v \setminus \{0\}$, $1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \nsubseteq (K^\times)^p$.

We can already check that this is of first-order: $K = K(p)$ can be expressed by saying that $K = K^p$ or $K = K^{(p)}$, p-henselianity of a valuation and a field are of first-order as seen before[2], and ring-properties of the residue field as well as ordered-group-properties of the value group can be expressed by interpretability of those structures in the valued field.

4.2 ψ_p characterises v_K^p

The next 5 lemmas will be a long serie of calculations, grouping them together will yield the result.

Lemma 4.2.1. *Let (K, v) be a valued field, let $K \vDash T_p$, and suppose:*

1. *vK has no non-trivial convex p-divisible subgroup, or*

2. *$\mathrm{ch}(Kv) = p$ and Kv is not perfect.*

Then for any non-trivial proper coarsening w of v, we have $Kw \neq Kw(p)$.

Proof. Let w be a proper coarsening of v and let $\Delta < vK$ be the corresponding non-trivial convex subgroup of vK, so we have $wK = vk/\Delta$, and $\overline{v} : Kw \to \Delta$ is a valuation with residue field $(Kw)\overline{v} = Kv$. We aim to find a Galois extension of Kw of degree p.

In case 1, we have $\Delta \neq p\Delta$. Thus there is $x \in Kw$ with $\overline{v}(x) \notin p\Delta$. Now if $\mathrm{ch}(Kw) \neq p$, then $\overline{\zeta_p} \neq 1$ is a p^{th}-root of unity in Kw, so $Kw[\sqrt[p]{x}]$ is a Galois extension of Kw of degree p.

On the other hand if $\mathrm{ch}(Kw) = p$, then we may assume $\overline{v}(x) < 0$ by possibly replacing x by x^{-1}. Consider the polynomial $X^p - X - x$, the

[2] This is true only when $p \neq 2$, we will see what can be done for $p = 2$ later.

roots of which can not be in Kw: if $\alpha^p - \alpha = x$, then $\overline{v}(\alpha) < 0$; therefore $\overline{v}(\alpha^p - \alpha) = p\overline{v}(\alpha) = \overline{v}(x)$. Now $Kw[\alpha]$ is a Galois extension of Kw of degree p.

In case 2, $(Kw)\overline{v} = Kv$ is not perfect. Thus we can choose some $\overline{a} \notin (Kv)^p$ and any corresponding $a \in \mathcal{O}_{\overline{v}}^{\times}$ is also not in $(Kw)^p$. If $\mathrm{ch}(Kw) \neq p$, then as before $Kw[\sqrt[p]{a}]$ is a Galois extension of Kw of degree p. If $\mathrm{ch}(Kw) = p$, take any $x \in \mathcal{M}_{\overline{v}}$ and consider the polynomial $X^p - X - ax^{-p}$, a root of which in Kw would satisfy $\overline{v}(\alpha) = -\overline{v}(x)$, which yields $(\alpha x)^p - a = \alpha x^p \in \mathcal{M}_{\overline{v}}$. In the residue field, we would then have $(\overline{\alpha x})^p + \overline{a}$, contradicting our choice of $\overline{a}$. Therefore any root α of the polynomial generates a Galois extension of degree p. $\square$

Corollary 4.2.2. *Let (K, v) be a p-henselian valued field containing ζ_p. Suppose $K \neq K(p)$, $\mathrm{ch}(Kv) \neq p$ and $Kv = Kv(p)$ hold, then:*

$$v = v_K^p \Leftrightarrow vK \text{ has no non-trivial } p\text{-divisible subgroup.}$$

Proof. Right-to-left follows immediately from lemma 4.2.1 item 1: if vK has no non-trivial p-divisible convex subgroup, then v has no proper coarsening with p-closed residue field, so to say v is the coarsest valuation with p-closed residue field; by definition, this means $v = v_K^p$.

Conversely if vK has a non-trivial p-divisible subgroup Δ, then the corresponding coarsening w of v has p-closed residue field: take $a \in Kw$. If $\overline{v}(a) \leqslant 0$, then replace it by a^{-1}. Now if $\overline{v}(a) > 0$, then by p-divisibility of Δ, $\overline{v}(a) = p\overline{v}(b)$ for some $b \in Kw$. So replacing a by ab^{-p} if necessary, we can assume $\overline{v}(a) = 0$. But then $X^p - \overline{a}$ has a (simple) root in $(Kw)\overline{v} = Kv$ since it is p-closed by assumption, and applying p-Hensel's lemma 2.2.2, we get a root of $X^p - a$ in Kw. Since $\zeta_p \in K$, any Galois extension of degree p is generated by p^{th}-roots, so Kw is p-closed; therefore v is not the coarsest p-henselian valuation with p-closed residue field. $\square$

Lemma 4.2.3. *Let (K, v) be a p-henselian valued field of equicharacteristic p with p-closed residue field. Then:*

$$v = v_K^p \Leftrightarrow \forall x \in \mathcal{M}_v \setminus \{0\}\,,\, x^{-1}\mathcal{O}_v \nsubseteq K^{(p)}.$$

Proof. If K is p-closed, then v_K^p is trivial and $K^{(p)} = K$. Therefore, if $v = v_K^p$ then v is trivial and $\mathcal{M}_v = \{0\}$, so the statement on the right reads "$\forall x \in \emptyset$, ..." and trivially holds; now for the converse take $x \in K \setminus \{0\}$, then obviously $x^{-1}\mathcal{O}_v \subseteq K^{(p)} = K$, so for the statement on the right side to hold, the only possibility is $\mathcal{M}_v = \{0\}$. Thus we can assume from now on $K \neq K(p)$.

From p-henselianity we can deduce that under the assumptions of the lemma, $\mathcal{O}_v \subseteq K^{(p)}$: take $a \in \mathcal{O}_v$ and consider $X^p - X - a$, which has a root in Kv. What the statement expresses if that $v = v_K^p$ iff no proper coarsening w of v satisfy $\mathcal{O}_w \subseteq K^{(p)}$.

First we show "$\Rightarrow$" by contradiction: suppose $\exists x \in \mathcal{M}_v \setminus \{0\}$ such that $x^{-1}\mathcal{O}_v \subseteq K^{(p)}$. $x^{-1}\mathcal{O}_v$ is an $\mathcal{O}_v$-fractional-ideal – an $\mathcal{O}_v$-submodule I of the fraction field of $\mathcal{O}_v$ (here K) such that there exists an $a \in \mathcal{O}_v$ with $aI \subseteq \mathcal{O}_v$. Furthermore, if $v(y) \leqslant v(x)$, then $y^{-1}\mathcal{O}_v \subseteq x^{-1}\mathcal{O}_v$.

Now let $\mathcal{N} = \bigcup_{x\in A} x^{-1}\mathcal{O}_v$, where $A = \left\{x \in \mathcal{M}_v \mid x^{-1}\mathcal{O}_v \subseteq K^{(p)}\right\}$. We claim that $\exists a \in K$ such that $v(a) > v(A)$: if not, then $\forall x \in K$, $\exists y \in A$ such that $v(y) \geqslant v(x)$; therefore $x^{-1}\mathcal{O}_v \subseteq y^{-1}\mathcal{O}_v \subseteq K^{(p)}$ and $K = K^{(p)}$, so $K = K(p)$.

$\mathcal{N}$ is an $\mathcal{O}_v$-fractional-ideal since $a\mathcal{N} \subseteq \mathcal{O}_v$, better still, it is the maximal one such that $\mathcal{O}_v \subsetneq \mathcal{N} \subseteq K^{(p)}$: for any $z \in K^{(p)} \setminus \mathcal{N}$, take $\mathcal{Z}$ any $\mathcal{O}_v$-fractional-ideal containing z. It must contain $z\mathcal{O}_v$, which is not contained in $K^{(p)}$ since $z \notin \mathcal{N}$, so $\mathcal{Z} \nsubseteq K^{(p)}$.

Let Γ be the convex hull of the subgroup of vK generated by $v(\mathcal{N} \setminus \mathcal{O}_v)$:

- Γ is non-trivial by assumption.
- Any $\gamma \in v(\mathcal{N}\setminus\mathcal{O}_v)$ is p-divisible in vK: take $x \in \mathcal{N}$ such that $v(x) = \gamma$, now since $\mathcal{N} \subseteq K^{(p)}$, there is a $y \in K$ such that $y^p - y = x$. We have $v(x) = v(y^p - y) < 0$, so $pv(y) = v(x) = \gamma$.
- Γ is p-divisible: let $\gamma \in \Gamma$, assume $\gamma < 0$. By definition there are a finite number of $n_i \in \mathbb{Z}$, $\alpha_i \in v(\mathcal{N} \setminus \mathcal{O}_v)$ such that:

  $$\sum n_i\alpha_i \leqslant \gamma < 0$$

  Take $\alpha = \min(\alpha_i)$ and $n = \sum n_i$; now $n\alpha \leqslant \gamma < 0$. γ lies in exactly one interval of the form $[(k+1)\alpha, k\alpha[$, therefore for some k, $\beta = \gamma - k\alpha \in [\alpha, 0[$. Now since $\alpha \in v(\mathcal{N})$ and $\alpha \leqslant \beta$, also $\beta \in v(\mathcal{N})$. By what we've seen, both α and β are p-divisible in vK, and $\gamma = \beta + k\alpha$ as well. Since Γ is convex, $\frac{\gamma}{p} \in \Gamma$.

Now we assume $v = v_K^p$ and aim towards a contradiction.

Since any coarsening of v has non p-closed residue field, $\mathcal{N}$ does not contain any coarsening of $\mathcal{O}_v$: if $\mathcal{O}_w \subseteq \mathcal{N} \subseteq K^{(p)}$, then $X^p - X - a$ has a root in K for any $a \in \mathcal{O}_w$, so it has a root in $\mathcal{O}_w$ since valuation rings are integrably closed, and therefore $X^p - X - \overline{a}$ has a root in Kw; thus $Kw(p) = Kw$ and w cannot be a proper coarsening of v.

We claim that Γ is of rank 1: take any $\{0\} < \Delta < vK$ convex, and let w be the associated proper coarsening of v. We know that $\mathcal{O}_w \nsubseteq \mathcal{N}$, so there is $z \in \mathcal{O}_w \setminus \mathcal{N}$, in particular, $z \notin \mathcal{O}_v$. Suppose there exists $x \in$

$\mathcal{M}_w$ such that $0 < v(x) \leqslant v(z^{-1})$, then $z^{-1}x^{-1} \in \mathcal{O}_v \subseteq \mathcal{O}_w$. But now $x^{-1} \in z\mathcal{O}_w \subseteq \mathcal{O}_w$, which contradicts the choice of $x \in \mathcal{M}_w$. This means that $v(z) \in \Delta = \{\gamma \in vK \mid 0 \leqslant \pm\gamma < v(x)\,\forall x \in \mathcal{M}_w\}$. Since $z \notin \mathcal{N}$, we know that $v(z) < v(y) < 0$ for any $y \in \mathcal{N} \setminus \mathcal{O}_v$, so $v(\mathcal{N} \setminus \mathcal{O}_v) \subset \Delta$; and by definition of Γ, we have $\Gamma \subseteq \Delta$.

Since Γ embeds in $\mathbb{R}$, we can fix $\alpha \in \mathbb{R}$ and consider the following set:

$$\mathcal{N}_\alpha = \{x \in K \mid v(x) \geqslant \alpha v(y) \text{ for some } y \in \mathcal{N}\}$$

It is an $\mathcal{O}_v$-fractional ideal which strictly contains $\mathcal{N}$ if $\alpha > 1$:

- Let $x \in \mathcal{N}$: if $v(x) \geqslant 0$ then $v(x) \geqslant \alpha v(1)$, and if $v(x) < 0$ then $v(x) \geqslant \alpha v(x)$.

- Since $v(\mathcal{N} \setminus \mathcal{O}_v) \subseteq \Gamma$, we know that $\gamma = \inf(v(\mathcal{N}))$ exists in $\mathbb{R}$. We also know that Γ is a p-divisible subgroup of $\mathbb{R}$, therefore it must be dense. This means that the interval $]\alpha\gamma, \gamma[$ contains an element of Γ, which is of the form $v(x)$. Now $x \in \mathcal{N}_\alpha$ but $x \notin \mathcal{N}$.

- $\mathcal{N}_\alpha$ is clearly an $\mathcal{O}_v$-module, and any $b \in \mathcal{O}_v$ such that $v(b) > -\alpha\gamma$ will verify $b\mathcal{N} \subseteq \mathcal{O}_v$. Such a b exists since Γ is dense.

Recall that by construction $\mathcal{N}$ is the maximal $\mathcal{O}_v$-fractional ideal such that $\mathcal{N} \subseteq K^{(p)}$. To get a contradiction, we take $\alpha \in]1, 2 - \frac{1}{p}[$ and prove that $\mathcal{N}_\alpha \subseteq K^{(p)}$:

Let $z \in \mathcal{N}_\alpha \setminus \mathcal{N}$, so there is $y \in \mathcal{N}$ such that $v(y) > v(z) \geqslant \alpha v(y)$. Note that $\alpha v(y) < 0$, so $v(y) < 0$. Now $0 > v(zy^{-1}) \geqslant (\alpha - 1)v(y) > v(y)$ since $\alpha < 2$. This means $zy^{-1} \in \mathcal{N} \setminus \mathcal{O}_v$, so $v(zy^{-1}) \in \Gamma$ is p-divisible: $v(zy^{-1}) = v(a^p)$, thus $v(za^{-p}) = v(y)$, which means $za^{-p} \in \mathcal{N}$. Finally, since $\mathcal{N} \subseteq K^{(p)}$, there is $b \in K$ such that $b^p - b = za^{-p}$, and we can write $z = (ab)^p - a^p b$, and we have:

$$\begin{aligned}
v(a^p b) &= v(a^p) + v(b) \\
&= v(zy^{-1}) + \tfrac{1}{p} v(b^p) \\
&= v(z) - v(y) + \tfrac{1}{p} v(za^{-p}) \\
&= v(z) - v(y) + \tfrac{1}{p} v(y) \\
&\geqslant (\alpha - 1 + \tfrac{1}{p}) v(y) \\
&\geqslant v(y) \in \mathcal{N}
\end{aligned}$$

Therefore $a^p b \in \mathcal{N} \subseteq K^{(p)}$, now since $v(ab) > v(a^p b)$ also $ab \in \mathcal{N} \subseteq K^{(p)}$, and $z = (ab)^p - ab + ab - a^p b$ is a sum of elements of $K^{(p)}$ which is stable by addition: $z \in K^{(p)}$, so $\mathcal{N}$ can't be maximal.

Lastly, we prove "$\Leftarrow$" by contraposition: suppose $v \neq v_K^p$, then by definition v_K^p is a proper coarsening of v with p-closed residue field. As done before for v, $\mathcal{O}_{v_K^p} \subseteq K^{(p)}$. Now take any $x \in \mathcal{M}_v \setminus \mathcal{M}_{v_K^p}$, in particular $x \in \mathcal{O}_{v_K^p}^\times$ and:

$$x^{-1}\mathcal{O}_v \subseteq x^{-1}\mathcal{O}_{v_K^p} = \mathcal{O}_{v_K^p} \subseteq K^{(p)}$$

which means $\exists x \in \mathcal{M}_v \setminus \{0\}\,,\, x^{-1}\mathcal{O}_v \subseteq K^{(p)}$. $\square$

The cases $\mathrm{ch}(Kv) \neq p$ and equicharacteristic p have been taken care in the previous lemmas, but the most tedious case of mixed characteristic $(0,p)$ is yet to be dealt with; this will require two lemmas.

Lemma 4.2.4 (Koenigsmann, [11, lemma 3.2])**.** *Let (K,v) be a p-henselian valued field of mixed characteristic $(0,p)$ containing ζ_p. Then for any $a \in \mathcal{O}_v$ we have:*

$$1 + (1-\zeta_p)^p a \in (K^\times)^p \Leftrightarrow \exists x \in Kv,\ x^p - x - \overline{a} = \overline{0}$$

Proof. The proof relies on a good choice of polynomial: if we have $f(X)$ such that $1 + (1-\zeta_p)^p a \in (K^\times)^p$ iff f has a zero in K and such that $\overline{f}(X) = X^p - X - \overline{a}$, the lemma holds by p-hensel's lemma.

We claim that the following polynomial is a good choice:

$$f(X) = \left(X + \frac{1}{1-\zeta_p}\right)^p - \left(\frac{1}{(1-\zeta_p)^p} + a\right)$$

Now $f(\alpha) = 0 \Leftrightarrow (1-\zeta_p)^p f(\alpha) = 0 \Leftrightarrow (\alpha(1-\zeta_p)+1)^p = 1 + (1-\zeta_p)^p a$. In order to obtain $\overline{f}$, we need to calculate coefficients of f:

$$\begin{aligned} f(X) &= \sum_{k=0}^{p}\left[\binom{p}{k} X^k \frac{1}{(1-\zeta_p)^{p-k}}\right] - \frac{1}{(1-\zeta_p)^p} - a \\ &= X^p + \sum_{k=2}^{p-1}\left[\frac{(p-1)!}{(p-k)!k!} X^k \frac{p}{(1-\zeta_p)^{p-1}}(1-\zeta_p)^{k-1}\right] + \frac{p}{(1-\zeta_p)^{p-1}} X - a \end{aligned}$$

It is still unclear what the residue of f is but believe it or not, we are almost here. First note that $\overline{\zeta_p} = \overline{1}$ since 1 is the only root of unity in characteristic p. Let $g(X)$ be the minimal polynomial of ζ_p over $\mathbb{Q}$: $g(X) = X^{p-1} + \cdots + 1 = \prod_{k=1}^{p-1}(1-\zeta_p^k)$. Now:

$$\begin{aligned} p = g(1) &= (1-\zeta_p)(1-\zeta_p^2)\cdots(1-\zeta_p^{p-1}) \\ \frac{p}{(1-\zeta_p)^{p-1}} &= \frac{1-\zeta_p}{1-\zeta_p} \times \frac{1-\zeta_p^2}{1-\zeta_p} \times \cdots \times \frac{1-\zeta_p^{p-1}}{1-\zeta_p} \\ &= (1+\zeta_p)(1+\zeta_p+\zeta_p^2)\cdots(1+\zeta_p+\zeta_p^2+\cdots+\zeta_p^{p-2}) \end{aligned}$$

Therefore $\frac{p}{(1-\zeta_p)^{p-1}}$ has residue $2 \times 3 \times \cdots \times p-1 = (p-1)!$, but since $\mathrm{ch}(Kv) = p$, $(p-1)! = -1$ in Kv. This implies $v(p) = (p-1)v(1-\zeta_p) > 0$, and we can look again at the coefficients of f:

- for $2 \leqslant k < p$, $v(\frac{(p-1)!}{(p-k)!k!}) \geqslant 0$ since it is an integer;
- $v(\frac{p}{(1-\zeta_p)^{p-1}}) = 0$;
- for $k \geqslant 2$, $(k-1)v(1-\zeta_p) = \frac{k-1}{p-1}v(p) > 0$.

Therefore coefficients in front of $X^2, X^3, \ldots, X^{p-1}$ all have positive valuation and are consequently null in the residue field. Since the coefficient in front of X is $\frac{p}{(1-\zeta_p)^{p-1}}$ which has residue -1, we have $\overline{f}(X) = X^p - X - \overline{a}$ and we conclude. $\square$

We now proove a final lemma very similar to lemma 4.2.3 but for the mixed characteristic case:

Lemma 4.2.5. *Let (K, v) be a p-henselian valued field of mixed characteristic $(0, p)$ containing ζ_p with residue field perfect and p-closed, and with no non-trivial convex p-divisible subgroup of its value group. Then:*

$$v = v_K^p \Leftrightarrow \forall x \in \mathcal{M}_v \setminus \{0\}, \, 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \nsubseteq (K^\times)^p$$

Proof. Once again, if $K = K(p)$ then v_K^p is trivial, and if $v = v_k^p$ then $\mathcal{M}_v = \{0\}$ and the statement on the right side reads "$\forall x \in \emptyset, \ldots$" and holds. Conversely, if v is non-trivial, then $p \in \mathcal{M}_v \setminus \{0\}$ and $v(p) < \frac{p}{p-1}v(p) = v((\zeta_p - 1)^p)$; so to say $p^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subseteq \mathcal{M}_v \subseteq K \setminus \{-1\}$, and $1 + p^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subseteq K^\times = (K^\times)^p$. We can thus assume $K \neq K(p)$.

"$\Rightarrow$": We assume $v = v_K^p$. Consider first the case where there exists a proper coarsening w of v such that $\mathrm{ch}(Kw) = p$. Since $v = v_K^p$, any proper coarsening $\overline{u}$ of $\overline{v}$ lifts to a proper coarsening u of v, and $(Kw)\overline{u} = Ku$ is not p-closed by definition of v_K^p. Likewise, any proper refinement of $\overline{v}$ has p-closed residue field. We then have (and this holds for any field K and valuations $\mathcal{O}_v \subseteq \mathcal{O}_w$):

$$v = v_K^p \Rightarrow \overline{v} = v_{Kw}^p$$

In our case, $(Kw, \overline{v})$ is a valued field of equicharacteristic p with p-closed residue field, so we can apply lemma 4.2.3 to it:

$$\overline{v} = v_{Kw}^p \Rightarrow \forall \overline{x} \in \mathcal{M}_{\overline{v}} \setminus \{0\}, \, \overline{x}^{-1}\mathcal{O}_{\overline{v}} \nsubseteq Kw^{(p)}$$

Given a $x \in \mathcal{M}_v \setminus \mathcal{M}_w$, we then know that for some $\overline{a} \in \mathcal{O}_{\overline{v}}$, $\overline{x}^{-1}a \notin Kw^{(p)}$, and by lemma 4.2.4 we have $1 + x^{-1}(1-\zeta_p)^p a \notin (K^\times)^p$. Doing this with $x = 1$ gives us an $a \in \mathcal{O}_v$ such that $1 + (1-\zeta_p)^p a \notin (K^\times)^p$. Finally, for $x \in \mathcal{M}_w \setminus \{0\}$, we have $1 + x^{-1}(\zeta_p - 1)^p xa \notin (K^\times)^p$, with $xa \in \mathcal{O}_v$; parsing everything together, we have:

$$v = v_K^p \Rightarrow \forall x \in \mathcal{M}_v \setminus \{0\}\,,\ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \nsubseteq (K^\times)^p$$

Thus in the case where a coarsening of v has residue characterisitc p, the proof of left-to-right is done.

Assume now that all coarsenings of v have residue characteristic 0. Then, we claim that $1+\mathcal{M}_v \nsubseteq (K^\times)^p$: consider the coarsening w of v corresponding to the maximal convex p-divisible subgroup of vK, which is non-trivial by assumption. We know that $\mathrm{ch}(Kw) = 0$, $\overline{v}$ is p-henselian, has p-divisible value group and perfect residue field. If $1 + \mathcal{M}_v \subseteq (K^\times)^p$, then also $1 + \mathcal{M}_{\overline{v}} \subseteq (Kw^\times)^p$. Note also that $\overline{\zeta_p} \neq \overline{1} \in Kw$, since otherwise the calculation of $w(\zeta_p - 1)$ in the proof of lemma 4.2.4 would yield $w(\zeta_p - 1) = \frac{w(p)}{p-1} = 0$, contradicting $\zeta_p - 1 \in \mathcal{M}_w$. Take now $a \in Kw$, by p-divisibility of $\overline{v}Kw$ we can find $b \in Kw$ such that $\overline{v}(ab^{-p}) = 0$. Since $(Kw)\overline{v}$ is perfect, $\overline{ab^{-p}} = \overline{c}^p$ for some $\overline{c} \in Kv$, and lifting it we have $ab^{-p} \in c^p(1 + \mathcal{M}_{\overline{v}})$ and thus $a \in Kw^p$. This means that w is a proper coarsening of v with p-closed residue field, contradicting $v = v_K^p$.

Now, we assume the following and aim for a contradiction:

$$\exists x \in \mathcal{M}_v,\ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subseteq (K^\times)^p$$

As before, $v(y) \leqslant v(x)$ implies $1+y^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subseteq (K^\times)^p$, and we can define $\mathcal{N} = \bigcup_{x \in A} x^{-1}\mathcal{O}_v$, where $A = \{x \in \mathcal{M}_v \mid 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subseteq (K^\times)^p\}$.

Let $a = -(\zeta_p - 1)^p$. Now $1 + a^{-1}(\zeta_p - 1)^p = 0 \notin (K^\times)^p$, therefore $a \notin A$ and any $x \in K$ with $v(x) \geqslant v(a)$ is also not in A. Thus any $y \in \mathcal{N}$ has value $v(y) > v(a^{-1})$, so $a\mathcal{N} \subset \mathcal{M}_v \subseteq \mathcal{O}_v$ and $\mathcal{N}$ is an $\mathcal{O}_v$-fractional-ideal.

Furthermore, $\mathcal{N}$ is the maximal $\mathcal{O}_v$-fractional-ideal such that $1 + (\zeta_p - 1)^p \mathcal{N} \subseteq (K^\times)^p$: for any $z \in (K^\times)^p \setminus \mathcal{N}$, take any $\mathcal{O}_v$-fractional-ideal $\mathcal{Z}$ containing it. $\mathcal{Z}$ must contain $z\mathcal{O}_v$, but $1 + (\zeta_p - 1)^p z\mathcal{O}_v \nsubseteq (K^\times)^p$ since $z^{-1} \notin A$, so $1 + (\zeta_p - 1)^p \mathcal{Z} \nsubseteq (K^\times)^p$.

Note also that since $1 + \mathcal{M}_v \nsubseteq (K^\times)^p$, we have $\mathcal{N} \subsetneq a^{-1}\mathcal{M}_v = (\zeta_p - 1)^{-p}\mathcal{M}_v$.

Let Γ be the convex hull of the subgroup of vK generated by $v(\mathcal{N} \setminus \mathcal{O}_v)$:

- Γ is non-trivial by assumption.
- Any $\gamma \in v(\mathcal{N} \setminus \mathcal{O}_v)$ is p-divisible: take $x \in \mathcal{N}$ such that $v(x) = \gamma < 0$, then since $1 + (\zeta_p - 1)^p \mathcal{N} \subseteq (K^\times)^p$ we have $1 + (\zeta_p - 1)^p x = a^p$ for some

$a \in K$, and since $\mathcal{N} \subsetneq (\zeta_p - 1)^{-p}\mathcal{M}_v$ we have $\overline{a}^p = 1 + \overline{(\zeta_p - 1)^p x} = 1 \in Kv$ and thus $\overline{a} = 1$ since $\mathrm{ch}(Kv) = p$. Hence, for some $b \in \mathcal{M}_v$:

$$1 + (\zeta_p - 1)^p x = (1+b)^p = \sum_{k=0}^{p} \binom{k}{p} b^k$$

Recall that $v(\zeta_p - 1) = \frac{v(p)}{p-1}$, therefore:

$$\begin{aligned}\min_{k=1,p}(\binom{k}{p} kv(b)) \leqslant v(\sum_{k=1}^{p} \binom{k}{p} b^k) &= v(x(\zeta_p - 1)^p) \\ &= v(x) + p\frac{v(p)}{p-1} < p\frac{v(p)}{p-1}\end{aligned}$$

$$\begin{aligned}\min_{k=1,p}(\frac{p!}{k!(p-k)!} kv(b)) &< p\frac{v(p)}{p-1} \\ \min_{k=1,p}(\frac{(p-1)!}{(k-1)!(p-k)!} v(b)) &< \frac{v(p)}{p-1} \\ v(b) = \min_{k=1,p}(\binom{p-1}{k-1} v(b)) &< \frac{v(p)}{p-1}\end{aligned}$$

This then yields $v(b^p) < \frac{p}{p-1} v(p) < v(p)$, therefore $v(b^p) < v(\binom{p}{k} v(b^k))$ since p divides the cofficient. Thus $v(x(\zeta_p - 1)^p) = v(b^p)$, which means γ is p-divisible.

- Γ is p-divisible: the argument in the proof of lemma 4.2.3 actually shows that any convex hull of a subgroup generated by a set is p-divisible as soon as the set of generators is p-divisible.

$\mathcal{N}$ does not contain any proper coarsening of v: suppose $\mathcal{O}_v \subsetneq \mathcal{O}_w \subseteq \mathcal{N}$. We know that $\mathrm{ch}(Kw) = 0$, therefore $w(\zeta_p - 1) = 0$, and:

$$1 + \mathcal{M}_v \subseteq 1 + \mathcal{O}_w = 1 + (\zeta_p - 1)^p \mathcal{O}_w \subseteq 1 + (\zeta_p - 1)^p \mathcal{N} \subseteq (K^\times)^p$$

Which as seen before contradicts $v = v_K^p$.

Following the proof of lemma 4.2.3, we have $\Gamma \leqslant \mathbb{R}$ and for $1 < \alpha \in \mathbb{R}$, the following is an $\mathcal{O}_v$-fractional ideal stricly containing $\mathcal{N}$:

$$\mathcal{N}_\alpha = \{x \in K \mid v(x) \geqslant \alpha v(y) \text{ for some } y \in \mathcal{N}\}$$

Recall that $(\zeta_p - 1)^p \mathcal{N} \subsetneq \mathcal{M}_v$. If $v((\zeta_p - 1)^{-p}) = \inf(v(\mathcal{N}))$, then any $x \in K$ with $v(x) > v((\zeta_p - 1)^{-p})$ would be in $\mathcal{N}$, and therefore $(\zeta_p - 1)^p \mathcal{N}$ would equal

$\mathcal{M}_v$. Therefore we take α such that $\alpha > 1$, $\alpha < 2 - \frac{1}{p}$ and $\alpha < \frac{v((\zeta_p - 1)^{-p})}{\inf(v(\mathcal{N}))}$, the later being strictly bigger than 1. This yields $\mathcal{N}_\alpha \subseteq (\zeta_p - 1)^{-p}\mathcal{M}_v$, and we aim to contradict the maximality of $\mathcal{N}$ by proving $1 + (\zeta_p - 1)^p \mathcal{N}_\alpha \subseteq (K^\times)^p$.

Let $z \in \mathcal{N}_\alpha \setminus \mathcal{N}$, so there is some $y \in \mathcal{N}$ with $0 > v(y) > v(z) \geqslant \alpha v(y)$. Then $0 > v(zy^{-1}) \geqslant (\alpha - 1)v(y) > v(y)$, thus $zy^{-1} \in \mathcal{N} \setminus \mathcal{O}_v$. Therefore by p-divisibility of Γ there is $a \in K \setminus \mathcal{O}_v$ such that $v(zy^{-1}) = v(a^p)$, which gives $v(za^{-p}) = v(y)$, so $za^{-p} \in \mathcal{N} \setminus \mathcal{O}_v$, and there is $b \in \mathcal{M}_v$ such that:

$$\begin{aligned} 1 + za^{-p}(\zeta_p - 1)^p &= (1 + b)^b \\ z(\zeta_p - 1)^p &= a^p(b^p + \cdots + pb) \\ 1 + z(\zeta_p - 1)^p &= 1 + (ab)^p + \cdots + pa^p b \end{aligned}$$

and as before $v(b^p) = v(za^{-p}(\zeta_p - 1)^p)$. Note also that $z(\zeta_p - 1)^p \in \mathcal{M}_v$ thanks to our choice of α, so $(ab)^p \in \mathcal{M}_v$ and ab also. We will first finish the proof modulo the following claim:

$$pa^p b \in (\zeta_p - 1)^p \mathcal{N} \tag{Claim}$$

This implies the following:

$$\begin{aligned} 1 + z(\zeta_p - 1)^p &= 1 + (ab)^p + \cdots + pa^p b \\ &= (1 + ab)^p \underbrace{-pab - \cdots - p(ab)^{p-1}}_{\in \mathcal{M}_v} + \underbrace{pa^p b^{p-1} + \cdots + pa^p b}_{\in pa^p b \mathcal{O}_v} \\ &\in (1 + ab)^p + \mathcal{M}_v + pa^p b \mathcal{M}_v \subseteq (1 + ab)^p + pa^p b \mathcal{M}_v \\ &\qquad\qquad\qquad\qquad\qquad\quad \subseteq (1 + ab)^p + (\zeta_p - 1)^p \mathcal{N}\mathcal{M}_v \\ &\qquad\qquad\qquad\qquad\qquad\quad \subseteq (1 + ab)^p + (\zeta_p - 1)^p \mathcal{N} \\ &\qquad\qquad\qquad\qquad\qquad\quad \subseteq (1 + ab)^p (1 + (\zeta_p - 1)^p \mathcal{N}) \\ &\qquad\qquad\qquad\qquad\qquad\quad \subseteq (K^\times)^p \end{aligned}$$

Since this holds for any $z \in \mathcal{N}_\alpha$, we have $1 + (\zeta_p - 1)^p \mathcal{N}_\alpha \subseteq (K^\times)^p$, contradicting the minimality of $\mathcal{N}$.

Remains the claim to prove. Recall that we have :

$$v(b^p) = v(za^{-p}(\zeta_p - 1)^p)$$

We first aim to prove $a^p b \in (\zeta_p - 1)\mathcal{N}$. Since $y \in \mathcal{N}$, we just need to show $v(a^p b(\zeta_p - 1)^{-1}) > v(y)$:

$$\begin{aligned} v(a^p b(\zeta_p - 1)^{-1}) &= pv(a) + v(b) - v(\zeta_p - 1) \\ &= pv(a) + \frac{1}{p}v(z) - v(a) \\ &= (p - 1)v(a) + \frac{1}{p}v(z) \end{aligned}$$

We chose y, z and α such that $0 > v(y) > v(z) \geqslant \alpha v(y) > (2 - \frac{1}{p})v(y)$. This yields $v(z^p) > v(y^{2p-1})$, and from that:

$$\begin{aligned} v((zy^{-1})^{p-1}z) = v(y^{1-p}) + v(z^p) &> v(y^p) \\ v((a^p)^{p-1}z) &> v(y^p) \\ (p-1)v(a^p) + v(z) &> pv(y) \\ (p-1)v(a) + \frac{1}{p}v(z) &> v(y) \end{aligned}$$

Hence we have $a^p b \in (\zeta_p - 1)\mathcal{N}$. Finally, we write $a^p b = (\zeta_p - 1)x$ for some $x \in \mathcal{N}$. Now:

$$\begin{aligned} pa^p b &= p(\zeta^p - 1)^p(\zeta^p - 1)^{p-1}x \\ v(pa^p b(\zeta^p - 1)^{-p}) &= v(p) - (p-1)v(\zeta^p - 1) + v(x) \\ &= v(p) - (p-1)\frac{v(p)p-1}{+}v(x) \\ &= v(x) \end{aligned}$$

Thus $pa^p b(\zeta^p - 1)^{-p} \in \mathcal{N}$ and the claim is proven. $\square_{\Rightarrow}$

"$\Leftarrow$": We assume $\forall x \in \mathcal{M}_v \setminus \{0\}\,,\ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \nsubseteq (K^\times)^p$, so $\forall x \in \mathcal{M}_v$ there is $a \in \mathcal{O}_v$ such that $1 + (\zeta_p)x^{-1}a \notin (K^\times)^p$. We first suppose that there exists a proper coarsening w of v with residue characteristic p. Then, lemma 4.2.4 tells us that $\forall x \in \mathcal{M}_v \setminus \mathcal{M}_w$, $1 + (\zeta_p)x^{-1}a \notin (K^\times)^p \Rightarrow \overline{x}^{-1}\overline{a} \notin Kw^{(p)}$, so we have:

$$\forall \overline{x} \in \mathcal{M}_w \setminus \{0\}\,,\ \overline{x}^{-1}\mathcal{O}_{\overline{v}} \nsubseteq Kw^{(p)}$$

Now by lemma 4.2.3 applied to $(Kw, \overline{v})$, we have $\overline{v} = v^p_{Kw}$, which then yields $v = v^p_K$.

Therefore we can assume that all proper coarsenings of v have residue characteristic 0. Let w be a proper coarsening of v. Since $p \in \mathcal{M}_v$, there is $a \in \mathcal{O}_v$ such that $1 + \frac{1}{p}(\zeta_p - 1)^p a \notin (K^\times)^p$. But $p \notin \mathcal{M}_w$ since w has residue characteristic 0, So $\mathcal{O}_v[\frac{1}{p}] \subseteq \mathcal{O}_w$, so $\frac{1}{p}(\zeta_p - 1)^p a \in \mathcal{O}_w$. But by p-hensel's lemma 2.2.2, we have $1 + \mathcal{M}_w \subseteq (K^\times)^p$, and therefore $\frac{1}{p}(\zeta_p - 1)^p a \notin \mathcal{M}_w$. Taking the residue, $\overline{\frac{1}{p}(\zeta_p - 1)^p a} \neq 0 \in Kw$, and it cannot have a p^{root}, otherwise we would lift it to K. This means $Kw \neq Kw(p)$, so v is the coarsest valuation with p-closed residue field: $v = v^p_K$. $\square$

We now have to parse every result together to understand why ψ_p characterizes v_K^p, meaning that if $K \vDash T_p$, then $(K, v) \vDash \psi_p$ iff $v = v_K^p$. Let us go through the sentence step by step:

1. If $K = K(p)$ then $\mathcal{O}_v = K$

In the case where K is p-closed, $(K, v) \vDash \psi_p$ iff v is trivial iff $v = v_K^p$.

2. And if $K \neq K(p)$ then:

   (a) $\mathcal{O}_v$ is a valuation ring of K, and

   (b) v is p-henselian, and

   (c) if $Kv \neq Kv(p)$, then Kv is not p-henselian

In the case where K and Kv are not p-closed, then v_K^p is a refinement of v, and in Kv, $\overline{v_K^p} = v_{Kv}^p$, thus $v = v_K^p$ iff $\overline{v_K^p}$ is trivial iff Kv is not p-henselian (since it is not p-closed).

   (d) And if $Kv = Kv(p)$, then:

      i. vK has no non-trivial p-divisible convex subgroup

By corollary 4.2.2, in the case where $\mathrm{ch}(Kv) \neq p$ we are done, so we can restrict ourselves to the case $\mathrm{ch}(Kv) = p$ in the next statement (note that this is a disjunction).

      ii. Or it has one and:

         A. $\mathrm{ch}(K) = p$ and $\forall x \in \mathcal{M}_v \setminus \{0\}, x^{-1}\mathcal{O}_v \nsubseteq K^{(p)}$

This is the equicharacteristic p case, handled in lemma 4.2.3.

         B. Or (K, v) is of mixed characteristic p and Kv is not perfect

This is thanks to lemma 4.2.1 item 2. This is not an equivalence: if Kv is not perfect (and all the previous assumptions) then $v = v_K^p$. Now if $v = v_K^p$, then either Kv is not perfect, or it is perfect and we fall in the next (and last) clause:

         C. Or (K, v) is of mixed characteristic p, Kv is perfect and $\forall x \in \mathcal{M}_v \setminus \{0\}, 1 + x^{-1}(\zeta_p - 1)^p\mathcal{O}_v \nsubseteq (K^\times)^p$.

This is the mixed characteristic case, handled in lemma 4.2.5.

4.3 When $p = 2$

Until there we conveniently dodged the case $p = 2$, for a very good reason: corollary 3.2.3 gives us a first-order-ring-sentence saying that a field is p-henselian, but it need not work when $p = 2$ for euclidean fields. The sentence ψ_p that we wrote precendentely would still characterize v_K^p even for $p = 2$, but now "Kv is 2-henselian" is not writable in first-order. However, we will still characterize a 2-henselian valuation; not quite v_K^2 but something close enough:

Definition 4.3.1. On a field K, if Kv_K^2 is not euclidean then we let $v_K^{2*} = v_K^2$; and if Kv_K^2 is euclidean then we let v_K^{2*} be the coarsest 2-henselian valuation with euclidean residue field.

This is well defined, since if a valuation has euclidean residue field then any refinement also; therefore v_K^{2*} is always a coarsening of v_K^2. We can now use ψ_2 to characterize v_K^{2*} for all $K \vDash T_2$; let ψ_2^* be the following valued-field-sentence:

- If Kv is not euclidean, then ψ_2, and
- If Kv is euclidean, then $\mathcal{O}_v$ is a 2-henselian valuation ring and no non-trivial convex subgroup of vK is 2-divisible.

We claim that if $K \vDash T_2$, then $(K, v) \vDash \psi_2^*$ iff $v = v_*^2$: consider first the case Kv_K^2 non euclidean, then $v = v_K^{2*}$ iff $v = v_K^2$ iff Kv is non euclidean and $(K, v) \vDash \psi_2$, the later being truly first-order.

Now in the case Kv_K^2 euclidean, $v = v_K^{2*}$ iff Kv is euclidean, v is a 2-henselian valuation and no coarsening of v have euclidean residue field. It remains to check that this is equivalent to the property of vK given above:

Lemma 4.3.2. *Let (K, v) be a 2-henselian valued field and suppose Kv euclidean, then:*

$$v = v_K^{2*} \Leftrightarrow vK \text{ has no non-trivial convex 2-divisible subgroup.}$$

Proof. Let $\Delta \leqslant vK$ be a convex subgroup, and denote w the corresponding coarsening, so that $\overline{v} : Kw \to \Delta$. We want to show that Kw is euclidean iff Δ is 2-divisible:

- Suppose Kw euclidean and let $\delta \in \Delta$. Take $x \in Kw$ such that $\overline{v}(x) = \delta$. Since Kw is euclidean, either x or $-x$ admit a square root; the image of which by $\overline{v}$ is the division of δ by 2.

- Suppose Δ 2-divisible, and let $x \in Kw$. Since $\overline{v}(x) \in \Delta$, there is $y \in Kw$ such that $\overline{v}(y^2) = \overline{v}(x)$, so $a = xy^{-2} \in \mathcal{O}_{\overline{v}^\times}$ and $\overline{a} \neq 0 \in Kv$. Since Kv is euclidean, either $\overline{a}$ or $-\overline{a}$ has a square root in Kv, so to say one polynomial $X^2 \pm \overline{a}$ has a simple root in Kv (since euclidean implies characteristic 0), and by 2-henselianity we lift it to a square root of $\pm a = \pm xy^{-2}$. Finally if -1 was a square in Kw, by taking the residue we would have a square root of -1 in Kv as well; so Kw is indeed euclidean.

Since by definition v_K^{2*} is the only valuation having no coarsening with euclidean residue field, we have the equivalence. $\square$

We can now apply Beth's definability theorem, and grouping everything together we have the following:

Theorem 4.3.3 (Jahnke, Koenigsmann)**.** *For any prime number $p \neq 2$, there is a $\emptyset$-ring-formula φ_p such that if $K \vDash T_p$ then $\varphi_p(K) = \mathcal{O}_{v_K^p}$; and for $p = 2$ there is a $\emptyset$-ring-formula φ_2 such that if $K \vDash T_2$, then $\varphi_2(K) = \mathcal{O}_{v_K^2}$ when Kv_K^2 is not euclidean and $\varphi_2(K) = \mathcal{O}_{v_K^{2*}}$ when Kv_K^2 is euclidean.*

5 Definability in extensions of $\mathbb{Q}_p$

Our journey will end with a proof that v_p, the p-adic valuation, is ring-definable in any algebraic extension of $\mathbb{Q}_p$. In here we don't care about parameters, more carefull constructions have to be done in order to get rid of them.

5.1 Explicit definitions

A very beautiful formula dating back to Julia Robinson can define v_p in $\mathbb{Q}_p$:

$$\varphi(x) : \exists y \; 1 + px^q = y^q$$

Here q is a prime number different from p. Indeed, if $v_p(x) < 0$ then $v_p(1 + px^q) = v_p(px^q) = qv_p(x) + 1$; hence $v(1 + px^q) \neq v(y^q)$ for all y and $\mathbb{Q}_p \nvDash \varphi(x)$. On the other hand, if $v_p(x) \geqslant 0$ then $v_p(px^2) > 0$ and $X^2 - (1 + px^2)$ has a root by Hensel's lemma, so $\mathbb{Q}_p \vDash \varphi(x)$.

This formula works mainly because there is an element of minimum positive valuation, namely p, and because there is a polynomial to which we can easily apply Hensel's lemma. Some extensions of $\mathbb{Q}_p$ will still have nice enough properties, so that they will still have an explicit valuation definition. Let $K/\mathbb{Q}_p$ be algebraic. We have:

$$\mathbb{Z} \subseteq v_p K \subseteq \mathbb{Q} \quad \& \quad \mathbb{F}_p \subseteq K v_p \subseteq \mathbb{F}_p^{\mathrm{alg}}$$

Extensions are *nice* when either $v_p K \neq \mathbb{Q}$ or $K v_p \neq \mathbb{F}_p^{\mathrm{alg}}$. In nice extensions, v_p is again ring-definable:

- If $v_p K \neq \mathbb{Q}$, take $t \in K$ such that $v(t) = \gamma > 0$ and is not q-divisible for some prime q. The following set is ring-definable:

  $$I = \{x \in K \mid \exists y \; 1 + tx^q = y^q\} = \{x \in K \mid \gamma + qv(x) > 0\}$$

  It is not quite $\mathcal{O}_{v_p}$ but it contains it. Consider its stabilisator:

  $$R = \{a \in K \mid aI \subseteq I\}$$

  R is a ring and contains $\mathcal{O}_{v_p}$, it is therefore a coarsening of it; it is non-trivial since $t^{-2} \notin R$. The only possibility is $R = \mathcal{O}_{v_p}$, which is thus ring-definable.

- If $K v_p \neq \mathbb{F}_p^{\mathrm{alg}}$, we take a monic polynomial $f \in \mathcal{O}_{v_p}[X]$ such that $\overline{f}$ has no root and $\overline{f}'$ is not zero. We claim:

  $$\mathcal{M}_{v_p} \subseteq \frac{1}{f(K)} - \frac{1}{f(K)} \subseteq \mathcal{O}_{v_p}$$

  Indeed, if $v_p(x) \geqslant 0$, then since $\overline{f}(\overline{x}) \neq 0$, we have $f(x) \in \mathcal{O}_{v_p}$. On the other hand, if $v_p(x) < 0$, then $v(f(x)) < 0$ and $\frac{1}{f(x)} \in \mathcal{M}_{v_p}$. Thus, as long as the residue field is not separably closed, we can always define a set between the valuation ring and the max ideal.

  In order to obtain $\mathcal{O}_{v_p}$ we need to add a ring-definable set T which contains a lift of every element of $K v_p$. If the latter is finite, we can just take lifts of its element as parameters. If it is infinite, then it is PAC, and the the following set works:

  $$T = \frac{1}{f(K)} \cdot \frac{1}{f(K)} \subseteq \mathcal{O}_{v_p}, \, \overline{T} = K v_p$$

  See [6, lem. 3.2] for a proof of the last fact.

Both previous definitions fail when $v_p K = \mathbb{Q}$ and $K v_p = \mathbb{F}_p^{\mathrm{alg}}$. When $K = \mathbb{Q}_p^{\mathrm{alg}}$, we know by minimality of algebraically closed fields that no definition can exist; however the defect of mixed characteristic fields means that the case $K \neq \mathbb{Q}_p^{\mathrm{alg}}$, $v_p K = \mathbb{Q}$ and $K v_p = \mathbb{F}_p^{\mathrm{alg}}$ could occur. These are the *wild* extensions of $\mathbb{Q}_p$, for which no explicit definition is known; yet we can still show that v_p is ring-definable.

5.2 Canonical p-henselian valuations on extensions of $\mathbb{Q}_b$

Let's first look at $\mathbb{Q}_b$ in itself, where b is a prime number. Since v_b is henselian it is in particular p-henselian for any p. It must therefore be comparable with the canonical p-henselian valuation (which is non-trivial since $\mathbb{Q}_b$ is henselian and not p-closed), and we have to look at two cases:

- If $\mathcal{O}_{v_b} \subseteq \mathcal{O}_{v^p_{\mathbb{Q}_b}}$, then there must be a convex subgroup of $v_b\mathbb{Q}_b$ corresponding to this coarsening; but since $v_b\mathbb{Q}_b = \mathbb{Z}$, the only possibility is $\mathcal{O}_{v^p_{\mathbb{Q}_b}} = \mathcal{O}_{v_b}$.
- If $\mathcal{O}_{v^p_{\mathbb{Q}_b}} \subseteq \mathcal{O}_{v_b}$, then $\mathcal{O}_{v^p_{\mathbb{Q}_b}}/\mathcal{M}_{v_b}$ is a valuation ring of $\mathbb{Q}_b v_b = \mathbb{F}_b$, which has no non-trivial valuation, so again $\mathcal{O}_{v^p_{\mathbb{Q}_b}} = \mathcal{O}_{v_b}$.

The argument works in the same manner for an algebraic extension L of $\mathbb{Q}_b$, since $\mathbb{Z} \subseteq v_b L \subseteq \mathrm{Div}(\mathbb{Z}) = \mathbb{Q}$ has no non-trivial convex subgroup, and $\mathbb{F}_b \subseteq Lv_b \subseteq \mathbb{F}_b^{\mathrm{alg}}$ has no non-trivial valuation; note however that if L happens to be p-closed then v_L^p would be trivial.

Therefore v_b is the canonical p-henselian valuation on any non p-closed algebraic extension of the b-adics, and is in particular definable in any such extension containing a p^{th}-root of unity.

5.3 There and Back again

In order to deal with arbitrary algebraic extensions, we will need to go up by adjoining a root of unity, and then down by interpreting this new field in the bottom one.

Let $L/\mathbb{Q}_b$ be algebraic, and $L \neq L^{\mathrm{alg}}$. Then there exists a finite algebraic extension of L of degree $n \geqslant 2$, which can be extended to a Galois extension N of degree at most $n!$. If p divides $[N : L]$, then $\mathrm{Gal}(N/L)$ has a p-Sylow subgroup S_p; denote F its fixed field. Now N/F is a Galois extension of p-power degree, therefore F is not p-closed, and F/L is finite. Consider $M = F[\zeta_p]$, M is still not p-closed since it is a finite extension of F (if $p = 2$ then we have to argue that $\mathbb{Q}_b$ is not orderable and therefore no extension of it can be euclidean), so ψ_p defines v_b on M. Finally, we interpret M in L (with coefficient of minimal poynmial of generators of M as parameters), and the restriction of v_b to L is therefore definable.

6 NIPity of extensions of $\mathbb{Q}_p$

As an immediate consequence of the definability of the valuation, we know that an algebraic extension K of $\mathbb{Q}_p$ is NIP as a pure field iff (K, v) is NIP as a valued field (except $\mathbb{Q}_p^{\text{alg}}$, since in it the valuation is not definable; but this doesn't matter since we know that both ACF and ACVF are NIP theories). NIP henselian valued fields have recently been described algebraically by Franziska Jahnke and Sylvy Anscombe in [1]. We will apply their result to our specific case to obtain a classification of NIP algebraic extensions of $\mathbb{Q}_p$.

6.1 NIP henselian valued fields

Theorem 6.1.1 (Jahnke-Anscombe, 2019)**.** *Let (K, v) be a henselian valued field. Then (K, v) is NIP iff the following holds:*

1. *Kv is NIP, and*
2. *either*
   (a) *(K, v) is of equicharacteristic and is either trivial or separably defectless Kaplansky, or*
   (b) *(K, v) has mixed characteristic $(0, p)$, (K, v_1)[3]is finitely ramified, and $(K_1, \overline{v})$ checks 2a, or*
   (c) *(K, v) has mixed characteristic $(0, p)$ and $(K_2, \overline{v})$ is defectless Kaplansky.*

If K is an algebraic extension of $\mathbb{Q}_p$, equipped with the p-adic valuation, then several simplifications occur in the theorem. We can obviously ignore the equicharacteristic case, and since we are in rank 1, the standard decomposition gives $v_1 = v$ and v_2 trivial, so $K_2 = K$ and $K_1 = Kv$. Let's define two notions appearing in this theorem:

Definition 6.1.2.

- A valued field (K, v) of residue characteristic $p > 0$ is *Kaplansky* if:
  1. vK is p-divisible,
  2. Kv is perfect,
  3. and Kv has no separable extension of degree divisible by p.
- (K, v) is *finitely ramified* if the interval $[0, v(p)] \subseteq vK$ is finite.

In our case, since $\mathbb{Z} \subseteq vK \subseteq \mathbb{Q}$, (K, v) is finitely ramified iff vK is isomorphic to $\mathbb{Z}$. Parsing these informations together:

[3]See section 3 for the definition and notations of the standard decomposition.

Corollary 6.1.3. *Let $K/\mathbb{Q}_p$ be algebraic and let v be the p-adic valuation on K. Then (K, v) is NIP if and only if the following holds:*

1. *Kv is NIP, and*

2. *either (b) $vK \simeq \mathbb{Z}$, or (c) (K, v) is defectless Kaplansky.*

We will reformulate this characterisation of NIP extensions of $\mathbb{Q}_p$ in somewhat more concrete terms.

A first easy case to consider is when Kv is finite. Then it is NIP, which takes care of 1. All finite fields have extensions of degree p, so (K, v) can't be Kaplansky. We must then have $vK \simeq \mathbb{Z}$ for K to check 2, so both the ramification and inertia degrees are finite, which is equivalent to having $K/\mathbb{Q}_p$ finite. Such a K immediately checks 1 and 2; it is also obviously NIP by interpratability of finite extensions. The following lemma is therefore not very insightful:

Lemma 6.1.4. *Let $K/\mathbb{Q}_p$ be algebraic with Kv finite. Then K is NIP iff $K/\mathbb{Q}_p$ finite.*

This tackle the finite case. Now if Kv is infinite, by 1 it must be separably closed, since infinite extensions of finite fields are PAC, and PAC not SC fields have IP [4]. So $Kv = \mathbb{F}_p^{\text{alg}}$. Remains for this field to check 2, which gives two distinct cases, and we then have the following case distinction:

1. Kv finite & $vK \simeq \mathbb{Z}$,

2. $Kv = \mathbb{F}_p^{\text{alg}}$ & $vK \simeq \mathbb{Z}$

3. $Kv = \mathbb{F}_p^{\text{alg}}$ & K defectless Kaplansky.

Since case 1 is already done, we just need to understand extensions of $\mathbb{Q}_p$ with residue field $\mathbb{F}_p^{\text{alg}}$.

We will study these extensions via their Galois groups.

6.2 Inertia and ramification groups

Let (K, v) be any valued field. Let $G = \text{Gal}(K^{\text{sep}}/K)$ be its absolute Galois group. Let's fix an extension of v to K^{sep} and denote it by v^{sep}. We will define several interesting closed subgroups of G with their corresponding extensions and list their properties without proving them. Details can be found in [5].

Definition 6.2.1 (closed subgroups of G of interest)**.**

- The *decomposition subgroup* G^h and the associated field extension K^h are defined as follow:

$$G^h = \{\sigma \in G \mid \sigma(\mathcal{O}_{v^{\mathrm{sep}}}) = \mathcal{O}_{v^{\mathrm{sep}}}\}$$

$$K^h = \mathrm{Fix}(G^h),\ v^h = v^{\mathrm{sep}}|_{K^h}$$

  (K^h, v^h) is called the *henselization* of K, hence the h.

- The *inertia subgroup* G^t and the associated *inertia extension* of K are defined as follow:

$$G^t = \{\sigma \in G \mid \sigma(x) - x \in \mathcal{M}_{v^{\mathrm{sep}}}\ \forall x \in \mathcal{O}_{v^{\mathrm{sep}}}\}$$

$$K^t = \mathrm{Fix}(G^t),\ v^t = v^{\mathrm{sep}}|_{K^t}$$

  The t stands for "träge".

- The *ramification subgroup* and the associated *ramification extension* of K are defined as follow:

$$G^v = \{\sigma \in G \mid \sigma(x) - x \in x\mathcal{M}_{v^{\mathrm{sep}}}\ \forall x \in K^{\mathrm{sep}}\}$$

$$K^v = \mathrm{Fix}(G^v),\ v^v = v^{\mathrm{sep}}|_{K^v}$$

  The v stands for "verzweigt".

Looking at the definitions, it is clear that these are ordered as follow:

$$G^v \subseteq G^t \subseteq G^h \subseteq G \qquad K \subseteq K^h \subseteq K^t \subseteq K^v \subseteq K^{\mathrm{sep}}$$

Let us study them in order:

Proposition 6.2.2 (henselization [5, thm. 5.2.2])**.** *(K^h, v^h) is a henselian valued field. It also uniquely embeds in any henselian extension of K. K^h is trivial iff (K, v) is already henselian. A priori K^h depends on the choice of v^{sep}, but these choices give extensions conjugate over K. (K^h, v^h) is an immediate extension of (K, v).*

Proposition 6.2.3 (inertia field [5, thm. 5.2.7])**.** *G^t is a normal subgroup of G^h, so K^t is a Galois extension of K^h. (K^t, v^t) is also a purely inertial extension of (K^h, v^h), in the following sense: if $K^h \subseteq L \subseteq M \subseteq K^t$ with M/L finite, then $[M : L] = [Mv : Lv]$.*[4] *We also have $v^tK^t = vK$ and $K^tv^t = (Kv)^{\mathrm{sep}}$. Finally, if an extension L/K^h is such that $(Kv)^{\mathrm{sep}} \subseteq Lv$, then already $K^t \subseteq L$.*

Proposition 6.2.4 (ramification field [5, thm. 5.3.3])**.** *G^v is a normal subgroup of G^t, so K^v is a Galois extension of K^t. (K^v, v^v) is a purely ramified extension of (K^t, v^t), in the following sense: if $K^t \subseteq L \subseteq M \subseteq K^v$ with M/L finite, then $[M : L] = [vM : vL]$. We also have $K^v v^v = (Kv)^{\mathrm{sep}}$ and $v^v K^v = \bigcup_{q \neq \mathrm{ch}(Kv)} \mathrm{Div}_q(vK)$, the q-divisible hull of vK for all q prime different form $\mathrm{ch}(Kv)$. If Kv is of characteristic 0 then it is the full divisible hull, and $K^v = K^{\mathrm{sep}} = K^{\mathrm{alg}}$. If Kv is of characteristic p then G^v is the unique p-Sylow subgroup of G^t.*

We summarize some of these informations in fig. 2.

$$\begin{tikzcd}
K^{\mathrm{sep}} \arrow[d, dash] & \mathrm{Div}(vK) \arrow[d, dash] & (Kv)^{\mathrm{alg}} \arrow[d, dash] \\
K^v \arrow[d, dash] & \bigcup_{q \neq \mathrm{ch}(Kv)} \mathrm{Div}_q(vK) \arrow[d, dash] & (Kv)^{\mathrm{sep}} \arrow[d, dash] \\
K^t \arrow[d, dash] & vK \arrow[d, dash] & (Kv)^{\mathrm{sep}} \arrow[d, dash] \\
K^h \arrow[d, dash] & vK \arrow[d, dash] & Kv \arrow[d, dash] \\
K & vK & Kv
\end{tikzcd}$$

Figure 2: Special extensions of a valued field and their corresponding value groups and residue fields.

The last field extension we will define and study is the complement of the ramification group, represented in fig. 3. Its existence is guaranteed by the following theorem, proved in [12]:

Theorem 6.2.5 (Kuhlmann, Pank, Roquette)**.** *Let (K, v) be a valued field and fix an extension of v to the separable closure. Then there exists at least one G^h-complement of G^v, a closed subgroup $G^k \subseteq G^h$ such that $G^k G^v = G^h$ and $G^k \cap G^v = \{\mathrm{id}\}$. Denoting $K^k = \mathrm{Fix}(G^k)$, we then have $K^k K^v = K^{\mathrm{sep}}$ and $K^k \cap K^v = K^h$.*

Note that this theorem states existence of such complements, but a priori not uniqueness. A lot of these complements could exist. Complements

[4]Since v^h is henselian, any extension of K^h is canonically associated with a unique valuation.

are better understood via diagrams drawings, see fig. 3. In these drawings anything going up (straight or slanted) is a field extension, and there will be a lot of "diamonds":

$$\begin{tikzcd}[column sep=small, row sep=small] & AB & \\ B \arrow[ur, dash] & & A \arrow[ul, dash] \\ & A \cap B \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Anytime such a diamond appear, it will be drawn such that the bottom vertex is the intersection of the left and right vertices, and the top vertex is their compositum.

$$\begin{tikzcd} & K^{\mathrm{sep}} & \\ K^k \arrow[ur, dash] & & K^v \arrow[ul, dash] \\ & K^h \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Figure 3: Complement of ramification group

6.3 Special extensions of $\mathbb{Q}_p$

We can now apply this to $\mathbb{Q}_p$. Since it is henselian, we know $G^h = G$ and $\mathbb{Q}_p^h = \mathbb{Q}_p$. We also have $\mathbb{F}_p^{\mathrm{sep}} = \mathbb{F}_p^{\mathrm{alg}}$, and $\mathbb{Q}_p^{\mathrm{sep}} = \mathbb{Q}_p^{\mathrm{alg}}$. We conclude that an algebraic extension $K/\mathbb{Q}_p$ has residue $\mathbb{F}_p^{\mathrm{alg}}$ iff $\mathbb{Q}_p^t \subseteq K$. To be NIP, such an extension still needs to check condition 2.

Lemma 6.3.1. *Let $K/\mathbb{Q}_p$ be algebraic with $Kv = \mathbb{F}_p^{\mathrm{alg}}$. Then $vK \simeq \mathbb{Z}$ iff $K/\mathbb{Q}_p^t$ is finite.*

This tackle the case 2b. The last remaining case is 2c, when (K, v) is defectless Kaplansky. In our case, since Kv is already algebraically closed, we need only to worry about the value group. Furthermore, we need to make sure that (K, v) is defectless. Looking at $\mathbb{Q}_p^v$, we see that its value group is everything but p-divisible, and has no reason to be defectless. On the other hand, complements of $\mathbb{Q}_p^v$ are exactly in the inverse situation, so they should have p-divisible value group and no defect. More precisely, we apply theorem 6.2.5 to $\mathbb{Q}_p^t$ and find $(\mathbb{Q}_p^t)^k$ such that $(\mathbb{Q}_p^t)^k\mathbb{Q}_p^v = \mathbb{Q}_p^{\mathrm{alg}}$ and

$\mathbb{Q}_p^v \cap (\mathbb{Q}_p^t)^k = \mathbb{Q}_p^t$. We claim that any extension of any complement $(\mathbb{Q}_p^t)^k$ is defectless Kaplansky, and that any defectless Kaplansky extension of $\mathbb{Q}_p^t$ will contain a complement:

Lemma 6.3.2. *Let $K/\mathbb{Q}_p$ be algebraic with $Kv = \mathbb{F}_p^{\mathrm{alg}}$ – so $\mathbb{Q}_p^t \subseteq K$. Then the following are equivalent:*

1. *K contains some $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$,*
2. *$K\mathbb{Q}_p^v = \mathbb{Q}_p^{\mathrm{alg}}$,*
3. *K is defectless Kaplansky.*

Proof.

1 ⇒ 2 This is by definition of a complement. This implication is not needed to prove the lemma since the route we're taking is $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

1 ⇒ 3 Let $(\mathbb{Q}_p^t)^k$ be a $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$ such that $(\mathbb{Q}_p^t)^k \subseteq K$. Let L/K be finite, and write $L = K(a)$, where $a \in \mathbb{Q}_p^{\mathrm{alg}}$. Let $\bar{b}$ be the tuple of coefficients of the minimal polynomial of a over K. Take $K' = (\mathbb{Q}_p^t)^k(\bar{b})$ and $L' = K'(a)$, then $[L' : K'] = [L : K] = n$ and $K'/(\mathbb{Q}_p^t)^k$ is finite; denote its degree by m. Finally, since a and $\bar{b}$ lie in $\mathbb{Q}_p^{\mathrm{alg}} = (\mathbb{Q}_p^t)^k\mathbb{Q}_p^v$, we might wright them as elements of $(\mathbb{Q}_p^t)^k(\mathbb{Q}_p^v)$. Take $\bar{c}$ any finite tuple of $\mathbb{Q}_p^v$ containing every element of $\mathbb{Q}_p^v$ appearing in a and $\bar{b}$. Then $L' \subseteq (\mathbb{Q}_p^t)^k(\bar{c})$, and let N be the degree of $\bar{c}$ over $(\mathbb{Q}_p^t)^k$ and l its degree over L'. We have:

$$N = [(\mathbb{Q}_p^t)^k(\bar{c}) : (\mathbb{Q}_p^t)^k] = [(\mathbb{Q}_p^t)^k(\bar{c}) : L'][L' : K'][K' : (\mathbb{Q}_p^t)^k] = nml$$

Those informations are compiled in the following diagram:

$$\begin{tikzcd}
L = K(\overline{a}) \arrow[d, dash, "n"'] & & (\mathbb{Q}_p^t)^k(\overline{c}) \\
K \arrow[d, dash, "\text{arbitrary}"'] & L' = K'(\overline{a}) \arrow[ur, dash, "l"] & \\
K' = K(\overline{b}) \arrow[d, dash, "m"'] \arrow[ur, dash, "n"'] & & \\
(\mathbb{Q}_p^t)^k \arrow[uuurr, dashed, dash, "N"'] & &
\end{tikzcd}$$

We will prove that N is not divisible by p, hence giving $n = [L : K]$ not divisible by p. Since $[L : K] = p^d[vL : vK]$, we will then have defectlessness ($d = 0$) and p-divisibility of vK.

Claim. $(\mathbb{Q}_p^t)^k$ and $\mathbb{Q}_p^v$ are linearly disjoint[5]over $\mathbb{Q}_p^t$.

Indeed, for any $\overline{x} \in (\mathbb{Q}_p^t)^k$ and $\overline{y} \in \mathbb{Q}_p^v$:

$$\begin{tikzcd}
(\mathbb{Q}_p^t)^k \arrow[ddr, dash] & & & & \mathbb{Q}_p^v \arrow[ddl, dash] \\
& & \mathbb{Q}_p^t(\overline{x}, \overline{y}) \arrow[dl, dash, "s'"'] \arrow[dr, dash, "r'"] & & \\
& \mathbb{Q}_p^t(\overline{x}) \arrow[dr, dash, "r"'] & & \mathbb{Q}_p^t(\overline{y}) \arrow[dl, dash, "s"] & \\
& & \mathbb{Q}_p^t & &
\end{tikzcd}$$

Now we have $r = p^d$ and s coprime with p. By definition $rs' = r's$, so p^d divides $r's$, thus it divides r'. Finally, $s' \leqslant s$, giving:

$$s' = \frac{r'}{p^d}s \leqslant s$$

Thus $s' = s$, $r' = r$, and we have linear disjointness.

Now let $\overline{d}$ be a tuple in $(\mathbb{Q}_p^t)^k$ containing coefficients of the minimal polynomials of elements in $\overline{c}$ over $(\mathbb{Q}_p^t)^k$. We know have by linear disjointness:

$$N = [(\mathbb{Q}_p^t)^k(\overline{c}) : (\mathbb{Q}_p^t)^k] = [\mathbb{Q}_p^t(\overline{d}, \overline{c}) : \mathbb{Q}_p^t(\overline{d})] = [\mathbb{Q}_p^t(\overline{c}) : \mathbb{Q}_p^t]$$

So in the following diagram, we know the top N and deduce the bottom N:

$$\begin{tikzcd}
(\mathbb{Q}_p^t)^k \arrow[ddr, dash] & & & & \mathbb{Q}_p^v \arrow[ddl, dash] \\
& & \mathbb{Q}_p^t(\overline{c}, \overline{d}) \arrow[dl, dash, "N"'] \arrow[dr, dash] & & \\
& \mathbb{Q}_p^t(\overline{d}) \arrow[dr, dash] & & \mathbb{Q}_p^t(\overline{c}) \arrow[dl, dash, "N"] & \\
& & \mathbb{Q}_p^t & &
\end{tikzcd}$$

[5]Many equivalent definitions of linear disjointness exist. Here, we say that L and M are linearly disjoint over $K \subseteq L \cap M$ iff anytime we have $K \subseteq L_0 \subseteq L$ and $K \subseteq M_0 \subseteq M$ with $[L_0 : K] = l$ and $[M_0 : K] = m$, then $[L_0M_0 : M_0] = l$ and $[L_0M_0 : L_0] = m$.

Hence N correspond to an extension inside $\mathbb{Q}_p^v$. We know those extensions have degree prime to p, thus p does not divide N and K is defectless Kaplansky.

3 ⇒ 2 Let K containing $\mathbb{Q}_p^t$ be defectless Kaplansky. Consider $L = K\mathbb{Q}_p^v$. It must have divisible vaue group and algebraically closed residue field, thus $\mathbb{Q}_p^{\text{alg}}/L$ must be an immediate extension. Now take $a \in \mathbb{Q}_p^{\text{alg}}$ and consider $L(a)/L$. It is finite and immediate, hence purely defect; so $[L(a) : L] = p^n$. Now consider K' which is obtained by adding to K the coefficients if the minimal polynomial of a over L. We have $[L(a) : L] = [K'(a) : K'] = p^n$, and $[K'(a) : K] = [K'(a) : K'][K' : K] = p^n[K' : K]$. But since K is defectless Kaplansky, no finite extension of K can have degree divisible by p and thus $n = 0$ and $a \in L$; so to say $L = \mathbb{Q}_p^{alg}$.

2 ⇒ 1 Let K containing $\mathbb{Q}_p^t$ be big enough to have $K\mathbb{Q}_p^v = \mathbb{Q}_p^{\text{alg}}$. In terms of Galois group, keeping the same notation as in section 6.2, we have that $H = \text{Gal}(\mathbb{Q}_p^{\text{alg}}/K)$ is a closed subgroup of G^t, and the "big engouh" condition on K yields $H \cap G^v = \{\text{id}\}$. Recall that G^v is the unique p-Sylow of G^t. Thus H is a p'-subgroup, meaning that its order is not divisible by p.

Fact. In a prosolvable group, p'-subgroups can be extended into p'-Hall-subgroup, and the later are G-complements of p-Sylow subgroups.

This is a reformulation of known results about profinite groups, details can be found in [13, sec. 2.3].

Since $G = \text{Gal}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ is a prosolvable group[6]and G^t is a normal and closed subgroup, G^t is also prosolvable, and we can extend H into a G^t-complement of G^v; denote it by G^k and let $(\mathbb{Q}_p^t)^k = \text{Fix}(G^k)$. Now $H \subseteq G^k$ yields $(\mathbb{Q}_p^t)^k \subseteq K$, and $(\mathbb{Q}_p^t)^k$ is indeed the wanted complement. □

We can now state the characterization of NIP extension of $\mathbb{Q}_p$:

Theorem 6.3.3 (NIPity of extensions of $\mathbb{Q}_p$)**.** *The class of all NIP algebraic extensions of $\mathbb{Q}_p$ is the disjoint union of the 3 following classes:*

1. *finite extensions of $\mathbb{Q}_p$,*

[6]This is a well-known fact, but it turns out to be quite hard to provide a good reference for it, or even to know when exactly was it first stated. An argument can be found in [7, prop. 7.2] for all p-adiccally closed fields, and a more elementary argument of algebraic flavour can be found in [2, cor. 3.9].

2. *finite extensions of* $\mathbb{Q}_p^t$,

3. *arbitrary extensions of* $(\mathbb{Q}_p^t)^k$, *where* $(\mathbb{Q}_p^t)^k$ *is any* $\mathbb{Q}_p^t$*-complement of* $\mathbb{Q}_p^v$.

Figure 4 shows where those NIP extensions lie compared with the usual $\mathbb{Q}_p \subseteq \mathbb{Q}_p^t \subseteq \mathbb{Q}_p^v \subseteq \mathbb{Q}_p^{\text{alg}}$ tower. Note that this is a bit of a misdirection, since there are many possible choices of $(\mathbb{Q}_p^t)^k$, but the picture represents only 1.

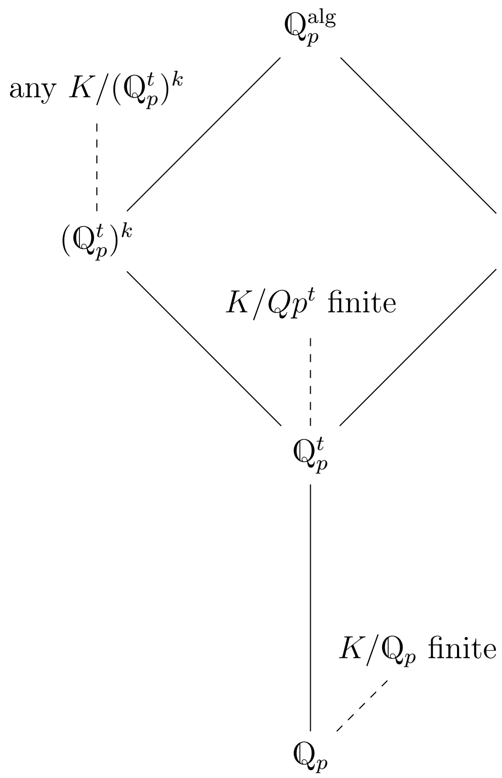

Figure 4: NIP algebraic extensions of $\mathbb{Q}_p$.

7 The road goes ever on

A lot of the machinery we applied to $\mathbb{Q}_p$ is much broader, so let's have a look at what can be done in more general settings. One reason why NIP valued fields are very much looked into is the following conjecture, attributed to Shelah:

Conjecture 7.1 (Field NIPity conjecture). *Let K be a NIP field, then K is either finite, separably closed, real closed, or admits a non-trivial henselian valuation.*

After what was done for $\mathbb{Q}_p$, several questions naturally arise: if the conjecture holds then there should be a way to pin-point this non-trivial henselian valuation just using the field-structure. Can we require it to be definable? The answer turns out to be yes, as we will show next.

Theorem 7.2. *Conjecture 7.1 is equivalent to the following statement:*

Let K be a NIP field, then K is either finite, separably closed, real closed, or admits a non-trivial definable henselian valuation v; (K, v) is then also NIP.

Proof. Clearly this statement is stronger than the original conjecture, so we just need to show that conjecture 7.1 implies it. So we assume conjecture 7.1 and we take K infinite, neither separably closed nor real closed.

Step 1: K is henselian iff v_K is non-trivial. This is clear by definition of the canonical henselian valuation v_K, keeping in mind that we took K not separably closed.

Step 2: K NIP $\Rightarrow$ Kv_K NIP. Let's make a case distinction on Kv_K. If it is separably closed or real closed, then it is NIP. Otherwise, we can go There and Back again as we did in $\mathbb{Q}_p$: we can find a prime p and a finite extension F/Kv_K such that F contains a p^{th}-root of unity and $F \neq F(p)$. Then we take a finite extension $(L, v)/(K, v_K)$ such that $F \subseteq Lv$, L contains a p^{th}-root of unity, and still $Lv \neq Lv(p)$.

Since Lv is not p-closed, $v_L^p \leqslant v$. Now we know that v_L^p is $\emptyset$-definable, so we can define $w = v_L^p|_K$ in K using coefficients of minimal polynomials of generators of L as parameters, and we know $w \leqslant v_K$.

Hence (K, w) is NIP since it is definable in K which is NIP, and (K, v_K) is externally definable in (K, w) since v_K correspond to a convex subgroup of wK; thus (K, v_K) is NIP.

Step 3: When K is NIP, a non-trivial coarsening of v_K is definable in K. This is true because we assumed conjecture 7.1. Indeed, v_K must then be non-trivial. Since we know also that Kv_K is NIP, we can apply conjecture 7.1 to it:

1. If Kv_K is finite, then we can define v_K in K, by applying the same method as in section 5.1 for finite residue fields.

2. If Kv_K is separably closed, then we go There and Back again. K is neither separably closed nor real closed, so we find $(L, v)/(K, v_K)$ finite containing a p^{th}-root of unity and not p-closed. Lv contains Kv_K, hence Lv is still separably closed, and thus $v_L^p \geqslant v$, so $w = v_L^p|_K \geqslant v_K$. w is definable in K and is non-trivial. Note: if $p = 2$ we then define v_L^{2*} instead of v_L^2, but since K is not real closed then either it is not Euclidean and v_L^{2*} is non-trivial, or it is Euclidean and we may assume $p \neq 2$.

3. If Kv_K is real closed, then we can define v_K^{2*} in K. It is a coarsening of v_K since v_K has Euclidean residue, and if K is not itself Euclidean then it is non-trivial. If K is Euclidean, then it must have extensions of odd degree, so it will have a non-p-closed extension for some odd p. We can then go There and Back again as in the separably closed case.

4. The remaining case can't happen; suppose that Kv_K is infinite, not separably closed and not real closed. Then by conjecture 7.1 it must admits a non-trivial henselian valuation. But since Kv_K is not separably closed, v_K is the finest henselian valuation on K, so Kv_K can't have any such valuation.

Thus, in conjecture 7.1, we may take v definable. Then (K, v) is still NIP. □

In a lot of cases we use a non-explicit formula to define this valuation, but we know that in the case where the residue is finite or when the value group is not divisible, we have explicit ways to define valuations. By carefully taking care of all cases, would it be possible to refine conjecture 7.1 by saying "if K is NIP then it is either separably closed, real closed, finite, or one of these formulas define a non-trivial henselian valuation"?

Another open question would be to adapt theorem 6.1.1 towards some form of NTP2 transfer. Several tools used by Anscombe and Jahnke in [1] can be extended into the NTP2 case, but not all; and the algebraic conditions underwhich transfer happens would need to be determined. A first step in this process could be to pinpoint NTP2 extensions of $\mathbb{Q}_p$.

References

[1] Sylvy Anscombe and Franziska Jahnke. Characterizing nip henselian fields, 2019.

[2] Nigel Boston. The proof of fermat's last theorem, 2003.

[3] Zoé Chatzidakis and Milan Perera. A criterion for p-henselianity in characteristic p. *Bull. Belg. Math. Soc. Simon Stevin*, 24(1):123–126, 03 2017.

[4] Jean-Louis Duret. Les corps faiblement algebriquement clos non separablement clos ont la propriete d 'independance. In Leszek Pacholski, Jedrzej Wierzejewski, and Alec J. Wilkie, editors, *Model Theory of Algebra and Arithmetic*, pages 136–162, Berlin, Heidelberg, 1980. Springer Berlin Heidelberg.

[5] Antonio J. Engler and A. Prestel. *Valued fields*. Springer, 2010.

[6] Arno Fehm. Existential $\emptyset$-definability of henselian valuation rings. *The Journal of Symbolic Logic*, 80(1):301–307, 2015.

[7] Dan Haran, Moshe Jarden, and Florian Pop. P-adically projective groups as absolute galois groups. *International Mathematics Research Notices*, 07 2005.

[8] Franziska Jahnke and Jochen Koenigsmann. Uniformly defining p-henselian valuations. *Annals of Pure and Applied Logic*, 166, 07 2014.

[9] Irving Kaplansky. Maximal fields with valuations. *Duke Math. J.*, 9(2):303–321, 06 1942.

[10] Jochen Koenigsmann. p-henselian fields. *manuscripta mathematica*, 87(1):89–99, Dec 1995.

[11] Jochen Koenigsmann. Encoding valuations in absolute galois groups. *Fields Institute Communications*, 33:107–132, 1999.

[12] Franz Viktor Kuhlmann, Matthias Pank, and Peter Roquette. Immediate and purely wild extensions of valued fields. *manuscripta mathematica*, 55(1):39–67, Mar 1986.

[13] Luis Ribes and Pavel Zalesskii. *Profinite Groups*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.