

Morley Rank, non-forking, canonical bases

Seminar on ample theories

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Source This talk contains material from classical textbooks, mostly from chapters 6,7 and 8 of Tent-Ziegler and chapter 6 and 8 from Marker.

Conventions We write lowercase letters for single elements or finite tuples. We write “ \subset ” for subsets – proper or not. We write “ Δ ” for symmetric difference of formulas: $\varphi \Delta \psi = (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$. We write “ \sqcup ” for disjoint unions of formulas, so $\varphi \sqcup \psi$ is the formula $\varphi \vee \psi$, but it is only defined on disjoint formulas. We write *we* for *many cups of coffee, the induced lack of sleep and I*.

Morley Rank

Ordinal-valued notion of dimension for formulas or definable sets.

Definition and basic properties

In a given structure \mathcal{M} , we define by induction:

- $\text{MR}_{\mathcal{M}}(\varphi) \geq 0$ iff φ is consistent;
- $\text{MR}_{\mathcal{M}}(\varphi) \geq \alpha + 1$ iff there are $(\varphi_i)_{i < \omega}$ disjoint, each implying φ , and each of $\text{MR} \geq \alpha$;
- $\text{MR}_{\mathcal{M}}(\varphi) \geq \lambda$ with λ a limit ordinal iff $\text{MR}(\varphi) \geq \alpha$ for all $\alpha < \lambda$.

Note that φ and φ_i are allowed to have different parameters.

Given a complete theory T , we define $\text{MR}_T(\varphi(x, a))$ to be $\text{MR}_{\mathcal{M}}(\varphi)$ for $\mathcal{M} \models T$ \aleph_0 -saturated and containing a .

Lemma 1. $\text{MR}_T(\varphi)$ is well defined, ie, it does not depend on the choice of an \aleph_0 -saturated model.

Proof.

- If \mathcal{M} is \aleph_0 -saturated and $\text{tp}_{\mathcal{M}}(a) = \text{tp}_{\mathcal{M}}(b)$, then $\text{MR}_{\mathcal{M}}(\varphi(x, a)) = \text{MR}_{\mathcal{M}}(\varphi(x, b))$.
 - If $\text{MR}=0$ it's clear, we proceed by induction.
 - Anytime $\varphi(x, a) = \varphi_1(x, c_1) \sqcup \dots \sqcup \varphi_n(x, c_n)$, by saturation we can find d_1, \dots, d_n such that $\text{tp}_{\mathcal{M}}(a, \bar{c}) = \text{tp}_{\mathcal{M}}(b, \bar{d})$.
- If $\mathcal{M} \preceq \mathcal{N}$ are both \aleph_0 -saturated, then $\text{MR}_{\mathcal{M}}(\varphi) = \text{MR}_{\mathcal{N}}(\varphi)$.
 - If $\text{MR}=0$ it's clear, we proceed by induction.
 - $\text{MR}_{\mathcal{M}}(\varphi) \leq \text{MR}_{\mathcal{N}}(\varphi)$ is clear.
 - In the other direction, we might have parameters from \mathcal{N} , but we can replace them by parameters from \mathcal{M} by saturation.
- Let $\mathcal{M} \models T$ containing parameters of φ . Then: $\mathcal{M} \begin{smallmatrix} \preceq \mathcal{N}_2 \\ \preceq \mathcal{N}_1 \end{smallmatrix} \preceq \mathcal{N}^*$

□

In the following, we assume that structures are at least \aleph_0 -saturated – equivalently, we work in a monster.

It is easy to check that $\text{MR}(\varphi \vee \psi) = \max(\text{MR}(\varphi), \text{MR}(\psi))$. By definition, we have $\text{MR}(\varphi) = 0$ iff $\varphi(M)$ is finite (and non-empty). If φ is inconsistent, we write $\text{MR}(\varphi) = -\infty$. Since the Morley rank of $\varphi(x, a)$ only depends on $\text{tp}(a)$ and there is no gap in the values of MR , we have:

Proposition 2. If $\text{MR}(\varphi) > (2^{|T|})^+$, then $\text{MR}(\varphi) \geq \alpha$ for all α ; we write $\text{MR}(\varphi) = \infty$.

Definition 3. We define an equivalence relation $\varphi \sim_{\alpha} \psi$ by $\text{MR}(\varphi \Delta \psi) < \alpha$.

A formula of $\text{MR} \alpha$ is α -strongly-minimal if for any ψ , either $\varphi \wedge \psi$ or $\varphi \wedge \neg \psi$ is of $\text{MR} < \alpha$.

If a formula is of $\text{MR} \alpha$, the maximal amount of definable subsets of $\varphi(M)$ of $\text{MR} \alpha$ is called the Morley Degree of φ .

Proposition 4. $\text{MD}(\varphi)$ is well-defined (if $\text{MR}(\varphi)$ is not $\pm\infty$): there is a decomposition $\varphi = \varphi_1 \sqcup \dots \sqcup \varphi_d$, with all φ_i α -strongly-minimal, unique up to \sim_{α} .

Proof. It's clear that if such a decomposition doesn't exist, φ has $\text{MR} > \alpha$. To prove uniqueness, take ψ α -strongly-minimal implying φ ; then there is exactly one φ_i such that $\varphi_i \wedge \psi$ is of rank α , and thus $\psi \sim_\alpha \varphi_i$. \square

- α -strongly-minimal $\Leftrightarrow \text{MR} = \alpha, \text{MD} = 1$
- $\text{MR}(\varphi) = 0 \Rightarrow \text{MD}(\varphi) = |\varphi(M)|$
- 0-strongly-minimal $\Leftrightarrow |\varphi(M)| = 1$
- 1-strongly-minimal \Leftrightarrow strongly-minimal as usual

Definition 5. For a type p , we define:

$$\text{MR}(p) = \min_{\varphi \in p} (\text{MR}(\varphi)), \text{ and } \text{MD}(p) = \min_{\varphi \in p, \text{MR}(\varphi) = \text{MR}(p)} (\text{MD}(\varphi))$$

We also write $\text{MR}(A/B) = \text{MR}(\text{tp}(A/B))$ and similarly for MD.

We have $\text{MR}(A/B) = 0$ iff $A \subset \text{acl}(B)$.

In strongly minimal theories

Recall that acl is a pregeometry on strongly-minimal theories:

- $A \subset B \Rightarrow \text{acl}(A) \subset \text{acl}(B)$
- $A \subset \text{acl}(A)$
- $\text{acl}(\text{acl}(A)) = \text{acl}(A)$
- $\text{acl}(A) = \bigcup_{A_0 \subset A \text{ finite}} \text{acl}(A_0)$
- $b \text{acl}(Ac) \setminus \text{acl}(A) \Rightarrow c \in \text{acl}(Ab)$

The last property, called Exchange, might fail outside of strongly minimal theories; the four others always hold.

Recall that a basis for A over B is a subset A' such that $\text{acl}(A'B) = \text{acl}(AB)$ and for any $X \subsetneq A'$, $\text{acl}(XB) \subsetneq \text{acl}(AB)$. $\dim(A/B)$ is the cardinal of a basis of A over B ; this is well defined.

We write $A \downarrow_C B$ if $\dim(a/C) = \dim(a/BC)$ for all finite $a \in A$.

Theorem 6. *In a strongly minimal theory, we have:**

$$\text{MR}(a_1, \dots, a_n/B) = \dim(a_1, \dots, a_n/B)$$

*here a_1 is a point, not a tuple.

For $n = 1$, either $\dim(a/B) = 0 \Leftrightarrow a \in \text{acl}(B) \Leftrightarrow \text{MR}(a/B) = 0$, or $\dim(a/B) = 1 \Leftrightarrow a \notin \text{acl}(B) \Leftrightarrow \text{MR}(a/B) \geq 1$. By strong minimality, any formula with 1 free variable is of $\text{MR} \leq 1$, so we are done.

The strategy for arbitrary n is the same: first we deal with the case $\dim < n$, then with the case $\dim = n$.

Lemma 7. *If $b \in \text{acl}(Ca)$, $\text{MR}(b/C) \leq \text{MR}(a/C)$.*

Proof. We work by induction on $\alpha = \text{MR}(a/C)$. If $\alpha = 0$, it is clear that $\text{MR}(b/C) = 0$.

We have $\text{MR}(b/Ca) = 0$, let $d = \text{MD}(b/Ca)$. We take $\psi_1(x) \in \text{tp}(a/C)$ of $\text{MR} \alpha$, and we take $\psi_2(a, y) \in \text{tp}(b/Ca)$ of $\text{MR} 0$ and $\text{MD} d$. We may assume $\mathcal{M} \models \forall x \exists^{\leq d} y \psi_2(x, y)$. Now let $\varphi(x, y) = \psi_1(x) \wedge \psi_2(x, y)$. We have:

$$\text{MR}(\exists y \varphi(x, y)) = \alpha \text{ and } |\varphi(a', M)| \leq d.$$

Consider $\chi(y) = \exists x \varphi(x, y)$. We will prove $\text{MR}(\chi) \leq \alpha$; since $\chi \in \text{tp}(b/A)$, this proves $\text{MR}(b/A) \leq \alpha$.

Let χ_i be an infinite family defining disjoint subsets of χ , say with parameters in C' . Let $\psi_i(x) = \exists y (\varphi(x, y) \wedge \chi_i(y))$. ψ_i implies $\exists y \varphi(x, y)$, and since any $d + 1$ of the ψ_i are disjoint, at least one of them must have $\text{MR} < \alpha$.

Take any b' realizing $\chi_i(y)$. Then by definition of χ there is a' realizing $\varphi(a', b')$. So $b' \in \text{acl}(C'a')$ and $\text{MR}(a'/C') \leq \text{MR}(\psi_i) < \alpha$, so by induction, $\text{MR}(b'/C') < \alpha$. Because this is true for any b' , we conclude $\text{MR}(\chi_i) < \alpha$. \square

This lemma allows us to only consider the case where the a_i are independent over B , that is, $\dim(a_1, \dots, a_n/B) = n$.

Proposition 8. *In a strong minimal theory, the type of n independent elements over a given subset is uniquely determined.*

Proof. In $\dim 1$ it's clear and has been done last week, prove the rest by induction. \square

Thus any formula on n free variables must have $\text{MR} \leq n$, so in particular, $\text{MR}(a_1, \dots, a_n/B) \leq n$.

Remains to prove that if $\dim(a_1, \dots, a_n/B) = n$, $\text{MR}(a_1, \dots, a_n/B) = n$.

- $\text{MR}(a_1, \dots, a_n/Ba_1) = n - 1$.
- Let $\psi \in \text{tp}(a_1, \dots, a_n/B)$. $\chi(\bar{x}, a_1) = \psi(x_1, \dots, x_n) \wedge x_1 = a_1$ has $\text{MR} \geq n - 1$.
- If $\text{tp}(a/B) = \text{tp}(a_1/B)$, $\text{MR}(\chi(\bar{x}, a)) = n - 1$.
- $\{\chi(\bar{x}, a) \mid a \equiv_B a_1\}$ is a disjoint family of subsets of ψ , so $\text{MR}(\psi) \geq n$.

\square_{thm}

Wanna fork?

From now on T is ω -stable; this is equivalent (in a countable language) to saying that the Morley Rank is never ∞ .

Definition 9.

- We write $A \downarrow_C B$ when $\text{MR}(a/C) = \text{MR}(a/BC)$ for any finite $a \in A$.
- We say that $\text{tp}(a/BC)$ forks over C when $a \not\downarrow_C B$.
- For $A \subset B$, $p \in S_n(A)$, $q \in S_n(B)$, we say that the extension $p \subset q$ is forking if $\text{MR}(p) > \text{MR}(q)$, or equivalently if q forks over A .
- $p \in S_n(C)$ is called stationary if for any $C \subset D$, p has a unique non-forking extension to D .

One can define forking in arbitrary theories but who has time for that? Certainly not us.

Lemma 10. *If $p \in S_n(A)$ has MD d and $A \subset \mathcal{M}$, then there are exactly d non-forking extensions of p in $S_n(\mathcal{M})$, and they are of MD 1.*

Proof. If $\varphi \in p$ realizes MR and MD of φ and $\varphi = \varphi_1 \sqcup \dots \sqcup \varphi_d$, then complete types over \mathcal{M} containing $p \cup \{\varphi_i\}$ are exactly the non-forking extensions of p . \square

Proposition 11. *If \mathcal{M} is κ -saturated and κ -homogeneous, any type forking over a subset A smaller than κ has at least κ many conjugates over A .*

No proof given.

Theorem 12 (Characterization of non-forking). *T is stable if and only if there is a special class of extensions of n -types, which we denote by $p \sqsubset q$, with the following properties:*

1. (Invariance) \sqsubset is invariant under $\text{Aut}(\mathcal{M})$,
2. (Local character) There is a cardinal κ such that for $q \in S_n(M)$ there is $C_0 \subset C$ of cardinality at most κ such that $q|_{C_0} \sqsubset q$.
3. (Weak Boundedness) For all $p \in S_n(A)$ there is a cardinal μ such that p has, for any $A \subset B$, at most μ extensions $q \in S_n(B)$ with $p \sqsubset q$.

If \sqsubset satisfies in addition:

4. (*Existence*) For all $p \in S_n(A)$ and $A \subset B$, there is $q \in S_n(B)$ such that $p \sqsubset q$,
5. (*Transitivity*) $p \sqsubset q \sqsubset r$ implies $p \sqsubset r$,
6. (*Weak Monotonicity*) $p \sqsubset r$ and $p \subset q \subset r$ implies $p \sqsubset q$,

then \sqsubset coincides with the non-forking relation.

If we have time, we will prove that in stable theories, those conditions characterize non-forking.

Canonical bases

Definition 13.

- $a \in \mathcal{M}$ is called a canonical parameter for a definable set $D \subset \mathcal{M}$ if for any $\sigma \in \text{Aut}(\mathcal{M})$, $\sigma(a) = a$ iff D is invariant under σ .
- $A \subset \mathcal{M}$ is called a canonical base for a type p if any $\sigma \in \text{Aut}(\mathcal{M})$ fixes A pointwise iff p is invariant under σ , that is: $\varphi(x, b) \in p \Leftrightarrow \varphi(x, \sigma(b)) \in p$.

Lemma 14. *Any definable set has an imaginary canonical parameter, that is, a canonical parameter in \mathcal{M}^{eq} .*

Proof. Write $X = \varphi(M, a)$. Define $x \sim y$ by $\varphi(M, x) = \varphi(M, y)$. $(a / \sim) \in \mathcal{M}^{eq}$ is a canonical parameter for X . \square

Note that the canonical parameter lies in $\text{dcl}^{eq}(a)$, also, EI is equivalent to saying each set has a (real) canonical parameter.

Lemma 15. *Any definable type has an imaginary canonical base.*

Proof.

- $B_\varphi = \{b \in \mathcal{M} \mid \varphi(x, b) \in p\}$ is definable by assumption
- $\sigma p = p$ iff $\sigma(B_\varphi) = B_\varphi$ for all φ
- Since B_φ is definable, it has a canonical parameter $a_\varphi \in \mathcal{M}^{eq}$
- $A = \{a_\varphi \mid \varphi \in \mathcal{L}\}$ is an imaginary canonical base for p .

\square

Proposition 16. *In an ω -stable theory, for any formulas $\varphi(x, a)$ of MR α and $\psi(x, y)$, the set $\{b \in \mathcal{M} \mid \text{MR}(\varphi(x, a) \wedge \psi(x, b)) = \alpha\}$ is a -definable.*

Proof. Write $\chi(x, b)$ for $\varphi(x, a) \wedge \psi(x, b)$.

- We might assume $\text{MD}(\varphi) = 1$.
- If $\text{MR}(\chi(x, c)) = \alpha$, there is a finite $X_c \subset \chi(\mathcal{M}, c)$ such that if $X_c \subset \psi(\mathcal{M}, b)$, $\text{MR}(\chi(x, b)) = \alpha$.
 - Chose $a_0 \in \chi(\mathcal{M}, c)$. If we can't find b_0 such that $a_0 \in \psi(\mathcal{M}, b_0)$ and $\text{MR}(\chi(x, b_0)) < \alpha$, then we can take $X_c = a_0$. Otherwise we take such a b_0 and continue by induction.
 - We have $\text{MR}(\chi(x, c) \wedge \bigwedge_{i \leq n} \neg \psi(x, b_i)) = \alpha$, so we can take a_{n+1} in there. Once again, if $X_c = \{a_0, \dots, a_n\}$ works, we're done; otherwise there is b_{n+1} such that $\{a_0, \dots, a_{n+1}\} \subset \psi(x, b_{n+1})$ and $\text{MR}(\chi(x, b_{n+1})) < \alpha$.
 - It has to stop because $\psi(a_i, b_j)$ holds iff $i \leq j$.
- Let $Y = \{X \subset \varphi(\mathcal{M}, a) \text{ finite} \mid X \subset \psi(\mathcal{M}, b) \Rightarrow \text{MR}(\chi(x, b)) = \alpha\}$ and let $\theta_X(y) = \bigwedge_{x \in X} \psi(x, y)$. Now $\text{MR}(\chi(x, b)) = \alpha$ iff $\bigvee_{X \in Y} \theta_X(b)$.
- We have the same result for $\neg \psi$.
- We move to a pair $(\mathcal{M}, \mathcal{M}^*)$ where $\mathcal{M} \preceq \mathcal{M}^*$ and \mathcal{M}^* is \mathcal{M} -saturated. We consider φ on \mathcal{M} and ψ on \mathcal{M}^* .
- By saturation we can finitize the disjunction, thus we have definability.
- If $\sigma(a) = a$, $\text{MR}(\chi(x, b)) = \text{MR}(\chi(x, \sigma(b)))$; thus we have a -definability.

□

Let $\varphi \in p$ realize MR and MD of p , then $p = \{\psi \mid \text{MR}(\varphi \wedge \psi) = \alpha\}$, thus:

Corollary 17. *In an ω -stable theory, any $p \in S_n(A)$ is definable over a finite $A_0 \subset A$, and thus has a finite imaginary canonical base in $\text{dcl}^{\text{eq}}(A_0)$.*

Theorem 18. *In an ω -stable theory, if $p \in S_n(A)$ and $\text{MD}(p) = 1$, then any non-forking extension q of p is A -definable.*

If $\text{MD}(p) > 1$, there is $a \in \text{acl}^{\text{eq}}(A)$ such that q is Aa -definable.

Proof. In degree 1, we have $q = \{\psi(x, b) \mid \text{MR}(\varphi \wedge \psi) = \alpha, b \in B\} \in S_n(B)$, so we might use the same definition than for p , needing only parameters appearing in φ .

In degree $d > 1$:

- Fix \mathcal{M} containing B , q (hence p) has a non-forking extension q^* to \mathcal{M} . q^* is of MD 1.
- Take $\varphi(x) \in p$ and $\psi(x, b) \in q^*$ realizing MR and MD of p and q^* . We can assume ψ implies φ .
- $X = \{c \mid \text{MR}(\psi(x, c)) = \alpha \text{ and } \forall d, \text{ if } \text{MR}(\psi(x, d)) = \alpha, \text{ then either } \text{MR}(\psi(x, c) \wedge \psi(x, d)) < \alpha \text{ or } \text{MR}(\psi(x, c) \wedge \neg\psi(x, d)) < \alpha\}$ is A -definable.
- $c \sim c'$: $\text{MR}(\psi(x, c) \wedge \psi(x, c')) = \alpha$ is a definable equivalence relation on X .
- $|X/\sim| \leq d$.
- For any $c \sim b$, $q = \{\chi \mid \text{MR}(\chi \wedge \psi(x, c)) = \alpha\}$, and q is Ac -definable.
- Because \sim has finitely many classes and is A -def, $a = (b/\sim) \in \text{acl}^{eq}(A)$, and q is Aa -def.

□

Corollary 19. *In an ω -stable theory, if $p \in S_n(\mathcal{M})$ doesn't fork over A , then p has a canonical base in $\text{acl}^{eq}(A)$. If $p|A$ is stationary, p has a canonical base in $\text{dcl}^{eq}(A)$.*

Proof. Fix $\varphi(x, a)$ realizing $\text{MR}(p)$, let $X = \{b \mid \varphi(x, b) \in p\}$. X is $\text{acl}(A)^{eq}$ (resp. A)-definable and $\sigma p = p$ iff $\sigma(X) = X$. Now X has a canonical parameter in $\text{dcl}^{eq}(\text{acl}^{eq}(A))$ (resp. $\text{dcl}^{eq}(A)$). □

If p has a canonical base A , we write $\text{cb}(p) = \text{dcl}^{eq}(A)$. This is well-defined.

Corollary 20. *In an ω -stable theory, $p \in S_n(\mathcal{M})$ doesn't fork over A iff $\text{cb}(p) \subset \text{acl}^{eq}(A)$.*

Reformulating, $a \perp_C B$ iff $p = \text{tp}(a/BC)$ doesn't fork over C iff $\text{cb}(p) \subset \text{acl}^{eq}(C)$.

Corollary 21. *In an ω -stable theory, If $A = \text{acl}^{eq}(A)$, $p \in S_n(A)$ is stationary.*

Proof. Let φ realise MR and MD of p . If MD=1, we're done. If not, I can write $\varphi(x) = \varphi_1(x, b_1) \sqcup \dots \sqcup \varphi_d(x, b_d)$. Let $q_i \in S_n(\mathcal{M})$ be the non-forking extension of p containing φ_i . $\text{cb}(q_i) \subseteq \text{acl}^{eq}(A) = A$, so we might assume $b_i \in \text{acl}^{eq}(A)$. But then $\varphi(x, b_i)$ or $\neg\varphi(x, b_i)$ must be in p already; so we must have $d = 1$. □

Proving non-forking characterization

Let \sqsubset check the conditions of theorem 12 and take $p \in S_n(A)$, $q \in S_n(B)$, $p \subset q$. If $p \sqsubset q$:

- By 3 (Boundedness) there is μ such that p has at most μ \sqsubset -extensions to $S_n(B)$.
- By proposition 11 we can take $B \subset \mathcal{M}$ such that any $r \in S_n(\mathcal{M})$ forking over A has at least μ conjugates over A .
- By 4 (Existence) and 5 (Transitivity) we can find $p \sqsubset r \in S_n(\mathcal{M})$.
- By 1 (Invariance) $p \sqsubset r'$ for any conjugate; since there can only be $< \mu$, they are non-forking.

The other direction needs the full power of canonical bases.

Lemma 22. *In an ω -stable theory, for $p \in S_n(A)$ and $\kappa > \max(|T|, |A|)$, in any \mathcal{M} strongly κ -homogeneous, all non-forking extensions of p to \mathcal{M} are conjugate.*

Proof. Let q_1, q_2 be extensions of p .

- In \mathcal{M}^{eq} , there is an A -automorphism of $\text{acl}^{eq}(A)$ sending $q_1|_{\text{acl}^{eq}(A)}$ to $q_2|_{\text{acl}^{eq}(A)}$.
- By strong homogeneity, the reduct of this to the base sort extends to an A -automorphism σ of \mathcal{M} , which in turn corresponds to an A -automorphism σ^{eq} of \mathcal{M}^{eq} .
- Now $\sigma^{eq}q_1$ is a non-forking extension of $q_2|_{\text{acl}^{eq}(A)}$; but by stationarity this must be q_2 .

□

Now we take $p \subset q$ non-forking:

- Take \mathcal{M} strongly κ -homogeneous for a large enough κ , take $q \subset r \in S_n(\mathcal{M})$ non-forking and $p \sqsubset r' \in S_n(\mathcal{M})$.
- We know $p \subset r'$ is non-forking.
- By the previous lemma r and r' are conjugate, so $p \sqsubset r$.
- By 6 (Monotonicity) we have $p \sqsubset q$.