Geometry of ACVF

Seminar on definable groups in metastable theories

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Conventions Throughout this talk, K will always denote a field, v its valuation, Γ its value group and k its residue field. We will also use K to denote the full valued field structure of K. We let letters like x denote seeminglessly variables or tuples of variables. We write *we* for *the terrible idea to have a seminar on monday, my ruined week-end and I.*

Classical results on ACVF

The following results are classical and will be provided without proofs, one can find more details in Martin Hills' lecture [2].

Definition 1. The three standard languages in which one can talk about valued fields in first-order are the following:

- $\mathcal{L}_{\text{div}}$, with only 1 sort for the field, $\mathcal{L}_{\text{div}} = \mathcal{L}_{\text{ring}} \cup \{|\}$, where the symbol $|$ is a binary relation, interpreted in the following way: $x|y \Leftrightarrow v(x) \leqslant v(y)$.
- $\mathcal{L}_{\Gamma}$, with 2 sorts, one for the field K equipped with $\mathcal{L}_{\text{ring}}$, and one for the value goup Γ equipped with $\mathcal{L}_{\text{OAG}}$. We also add a function symbol val, going from K to Γ, representing the valuation.
- $\mathcal{L}_{\Gamma k}$, with 3 sorts, one for the field K equipped with $\mathcal{L}_{\text{ring}}$, one for the value goup Γ equipped with $\mathcal{L}_{\text{OAG}}$, and one for the residue field k equipped with a copy of $\mathcal{L}_{\text{ring}}$. We also add the map Res, from $K \times K$ to k; it corresponds to the map $\text{Res}(a, b) = \text{res}(\frac{a}{b})$.

These 3 languages can be used interchangeably; notice that they are all bi-interpretable.

Theorem 2 (Robinson)**.** *The theory* ACVF *eliminates quantifiers in any of these 3 languages. Completions of* ACVF *are* $\mathrm{ACVF}_{0,0}$*,* $\mathrm{ACVF}_{0,p}$*, and* $\mathrm{ACVF}_{p,p}$*, where the index denotes the characteristics of the field and residue field.*

Recall that a definable set D is said to be *stably embedded* if any definable subset of D is D-definable. As a direct consequence of quantifier elimination, we have the following:

Corollary 3. Γ *and* k *are stably embedded in* $K \vDash \mathrm{ACVF}$*, in whichever of the 3 languages, and the induced structures on* Γ *and* k *are pure ordered groups and field structures. Moreover, they are orthogonal, meaning that any definable subset of* $\Gamma^n \times k^m$ *is a finite Boolean combination of rectangles* $A \times B$*, where* $A \subseteq \Gamma^n$ *is* Γ*-definable and* $B^m \subseteq k$ *is* k*-definable.*

Indeed, no function in the language has domain Γ or k, and anytime a term $\mathrm{val}(a)$ or $\mathrm{res}(b)$ occurs in a formula, we can replace it with parameters $\gamma = v(a)$ from Γ or $\bar{b}$ from k. Note that if $K \vDash \mathrm{ACVF}$, then Γ is o-minimal and k is strongly minimal.

Topology & C-minimality

Definable subsets of k and Γ are thus well understood. In order to describe more precisely the definable subsets of K, we need to study its canonical topology.

Definition 4 (Balls)**.** Let $a \in K$ and $\gamma \in \Gamma$. We define *open* and *closed balls of center* a *and radius* γ as follow:

- $B_{>\gamma}(a) = \{x \in K \mid v(x - a) > \gamma\}$
- $B_{\geqslant\gamma}(a) = \{x \in K \mid v(x - a) \geqslant \gamma\}$

Open balls form a base for a V-topology (which is in peculiar a topology) on K, usually denoted τ_v. Field operations are continuous for this topology.

Remark 5. Because of the ultrametric inequality, any element of a ball is a center of this ball. This implies that any two balls are either disjoint or nested. The latter fact remains true for generalized balls, see definition 6.

Definition 6 (Swiss cheese)**.** We call *generalized balls* any open or closed ball, as well as singletons (balls with infinite radius) and K itself (ball with negative infinite radius). A *swiss cheese* C is then defined to be a generalized ball with holes (see fig. 1):

$$C = B \setminus \bigcup_{i=1}^{n} B_i$$

Where B and $B_i \subsetneq B$ are generalized balls.

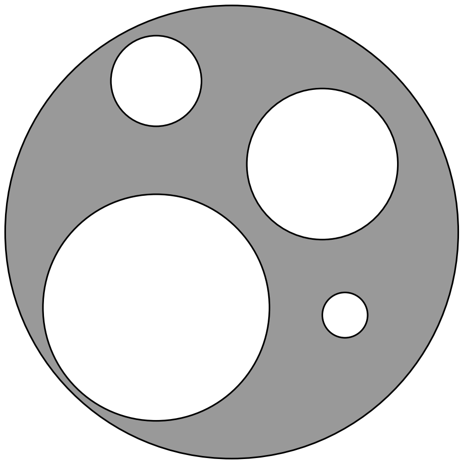

Figure 1: A swiss cheese.

Theorem 7 (Holly)**.** ACVF *is C-minimal, meaning that anything definable in a model of* ACVF *is a finite union of swiss cheeses.*

Corollary 8. $\mathrm{ACVF}_{p,q}$ *is NIP for any* $(p, q) = (\mathrm{ch}(K), \mathrm{ch}(k))$.

Proof. Let $\varphi(x, y)$ be any formula, with $|x| = 1$. By theorem 7, $\varphi(K, b)$ define a Boolean combination of generalized balls for any $b \in K^n$. By compactness, there is N such that for all b, $\varphi(K, b)$ is a Boolean combination of at most N generalized balls. Remark 5 tells us that generalized balls have VC-dimension 2, and since Boolean combinations of NIP formulas are NIP, φ is NIP. $\square$

About types

We're doing a small detour to study some types and their properties in ACVF, one can look into Will Johnson's notes [3] for more details.

Proposition 9. *The following types are definable:*

- *For any (not generalized) ball B, $p_B(x)$, the generic type of B, saying that $x \in B$ but $x \notin B'$ for any ball $B' \subsetneq B$. For small sets C, $p_B|C$ says the same for all C^{alg}-definable $B' \subsetneq B$.*

- *$p_k(x)$, the generic type of k, saying that $\mathrm{res}(x)$ is transcendental over k. For small sets C, $p_k(x)|C$ says that $\mathrm{res}(x) \notin \mathrm{res}(C)^{\mathrm{alg}}$.*

Lemma 10. *$a \vDash p_{\mathcal{O}}|C$ iff $\mathrm{res}(a) \vDash p_k|C$; so $p_{\mathcal{O}}$ is dominated along* res *and* $\mathrm{res}_* \, p_{\mathcal{O}} = p_k$.

Proof. Any subball of $\mathcal{O}$ is included in $\mathrm{res}^{-1}(\alpha)$ for some $\alpha \in k$. Thus, $a \vDash p_{\mathcal{O}}|C$ iff $a \notin \mathrm{res}^{-1}(\alpha)$ for all $\alpha \in \mathrm{res}(C)^{\mathrm{alg}}$. □

Lemma 11. *p_k is generically stable; so is $p_{\mathcal{O}}$ and any p_B for B a closed ball.*

Proof. We have $(a, b) \vDash p_k \otimes p_k|C \Leftrightarrow b \notin \mathrm{res}(C)^{\mathrm{alg}} \wedge a \notin (\mathrm{res}(C)b)^{\mathrm{alg}}$, thus it is clear that p_k commutes with itself and hence is generically stable. Since $\mathrm{res}_* \, p_{\mathcal{O}} = p_k$, and since any closed ball is definably in bijection with $\mathcal{O}$ (see the proof of lemma 15), p_B is generically stable for closed balls. □

Imaginaries

Definable subsets of ACVF are thus very well understood thanks to quantifier elimination. Interpretable subsets are more tricky, and in the standard languages ACVF does not eliminate imaginaries. For that we need to introduce new sorts. We will follow Haskell, Hrushovski and Macpherson's notation in [1] where they prove elimination of imaginaries.

Definition 12 (Geometric language)**.** We define the geometric language $\mathcal{L}_{\mathcal{G}}$ by adding to $\mathcal{L}_{\mathrm{div}}$ – or to any of the 2 other classical languages – the following:

- sorts S_n for $n \geqslant 1$ which are to be intrepreted as $\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$,

- sorts T_n for $n \geqslant 1$ which are to be intrepreted as $\bigcup_{s \in S_n} s/\mathcal{M}s$,

- relation symbols ϵ_n on $K^n \times S_n$, which correspond to the membership $\epsilon_n(a_1, \cdots, a_n, s) \Leftrightarrow (a_1, \cdots, a_n) \in s$,

- function symbols $\tau_n : T_n \to S_n$, which correspond to the projections $\tau_n(t) = s \Leftrightarrow t \in s/\mathcal{M}s$,

- partial function symbols $\nu_n : K^n \times S_n \to T_n$, which correspond to the projections $\nu_n(a, s) = a + \mathcal{M}s$ when $\epsilon_n(a, s)$,

- relation symbols ${}^*\varphi$ on $S_{n_1} \times \cdots \times S_{n_r} \times Y$ for every atomic formula $\varphi(x_1, \cdots, x_r, y)$, where the x_i are n_i^2-tuples – matrices – and y is a tuple with domain denoted Y; ${}^*\varphi(s_1, \cdots, s_r, b)$ will hold iff $\varphi(a_1, \cdots, a_r, b)$ holds for any generic resolution $(a_1, \cdots, a_r)$ of $(s_1, \cdots, s_r)$ over b.

Remark 13. The sort S_n correponds to the set of $\mathcal{O}$-lattices, and we have $\Gamma \simeq S_1 = K^\times/\mathcal{O}^\times$. The set T_n can be seen as the set of pairs (s, x), where $s \in S_n$ and $x \in s/\mathcal{M}s$. In this setting $\tau_n(s, a) = s$ and $\nu_n(a, s) = (s, \overline{a})$. Note also that $s/\mathcal{M}s \simeq s \otimes_{\mathcal{O}} k$, giving $T_1 \simeq \Gamma \times k \simeq RV$. These sets are all interpretable in ACVF.

Theorem 14 (Haskell, Hrushovski, Macpherson)**.** ACVF *eliminates quantifiers and imaginaries in $\mathcal{L}_{\mathcal{G}}$.*

In some sense, this elimination of imaginaries is optimal, as discussed in the original paper [1]. For example, adding sorts for all generalized balls is not enough. But some variant of these geometric sorts exist, sometimes simplifying a lot the proof of theorem 14, we can mention Will Johnson's proof in [3], using only one (much larger) family of sorts:

$$R_{n,l} = \{(s, v) \mid s \in S_n,\ v \text{ is a } l\text{-dimensionnal vector subspace of } \mathcal{M}s\}$$

This sort still contains (interpretations of) the usual geometric sorts.

Additional results

We turn again to Martin Hills' lecture [2] to list some results which will be usefull in the next talks.

Lemma 15. *If $K \vDash$ ACVF, then no definable function $f : k^n \to K^m$ or $f : \Gamma^n \to K^m$ can have infinite image.*

Proof.

- Suppose a definable function f has infinite image in K^m. Then one of $\pi_i \circ f$ must have infinite image. So it is enough to show the lemma for $m = 1$.

- If f is definable and has image in K, by C-minimality (theorem 7), its image must be a disjoint union of swiss cheeses.

- Any infinite cheese contains a closed ball.

To prove this point, let's define the radius of a generalized ball B in the following way:

$$\operatorname{rad}(B) = \begin{cases} \rho & \text{if } B = B_{\geqslant \rho}(a), \\ \rho^+ & \text{if } B = B_{>\rho}(a), \\ +\infty & \text{if } B = \{a\}, \\ -\infty & \text{if } B = K. \end{cases}$$

The radius lies in $\Gamma \sqcup \Gamma^+ \sqcup \{+\infty, -\infty\}$. For $\gamma \in \Gamma \cup \{\infty\}$, we write:

$$\gamma \geqslant \operatorname{rad}(B) \Leftrightarrow \begin{cases} \operatorname{rad}(B) = \rho & \& \quad \gamma \geqslant \rho, \\ \operatorname{rad}(B) = \rho^+ & \& \quad \gamma > \rho, \\ \operatorname{rad}(B) = +\infty & \& \quad \gamma = \infty, \\ \operatorname{rad}(B) = -\infty. \end{cases}$$

and we write $\gamma > \operatorname{rad}(B)$ for $\gamma \geqslant \operatorname{rad}(B) \wedge \gamma \neq \operatorname{rad}(B)$.

We now have for any generalized ball B that:

$$\begin{array}{lcl} B_{\geqslant \gamma}(c) \subseteq B & \Leftrightarrow & c \in B \text{ and } \gamma \geqslant \operatorname{rad}(B) \\ B_{\geqslant \gamma}(c) \supseteq B & \Leftrightarrow & B \cap B_{\geqslant \gamma}(c) \neq \emptyset \text{ and } \neg(\gamma > \operatorname{rad}(B)). \end{array}$$

Consider now an infinite cheese $C = B \setminus \bigcup_{i=1}^{n} B_i$. Since it is infinite, $\operatorname{rad}(B) \neq +\infty$. Let $c \in C$. We want to find γ such that $B_{\geqslant \gamma}(c) \subseteq B$ and $B_i \nsubseteq B_{\geqslant \gamma}(c)$. Regarding radiuses, it is enough to take $\gamma \geqslant \operatorname{rad}(B)$ and $\gamma > \operatorname{rad}(B_i)$. If no B_i has $+\infty$ for radius, then we can find such an γ. Otherwise, let $m = \max(\operatorname{rad}(B_j))$ for the B_j which are not a singleton, and let $m_i = v(c - a_i)$ for all $B_i = \{a_i\}$. Then m and m_i are not infinite, and $\gamma > \max(m, m_i)$ works: $B_{\geqslant \gamma}(c)$ is disjoint from the B_j because of their radiuses, and $a_i \notin B_{\geqslant \gamma}(c)$ since $v(c - a_i) < \gamma$.

- Any closed ball is in definable bijection with $\mathcal{O}$: $B_{\geqslant \gamma}(c) = a\mathcal{O} + c$ for any $a \in K^\times$ with $v(a) = \gamma$.

- The definable functions val and res are such that $\operatorname{val}(\mathcal{O})$ and $\operatorname{res}(\mathcal{O})$ are infinite.

- Take $f : k^n \to K$ of infinite image. Restrict its image to a chosen closed ball $B_{\geqslant \gamma}(c)$. Send $B_{\geqslant \gamma}(c)$ to $\mathcal{O}$ via the definable map $g : x \to \frac{x-c}{a}$. Now consider $\operatorname{val} \circ g \circ f|_{f^{-1}(B_{\geqslant \gamma}(c))}$. Its graph is a definable subset of $k^n \times \Gamma$, therefore by orthogonality it is a finite union of rectangles. But these rectangles must be of the form $D \times \{\gamma\}$, since it is the graph of a function. Then the image of this function is finite, contradicting what was said before.

- Same goes for $f : \Gamma^n \to K$ by composing it with the residue map.

$\square$

Proposition 16. $|K| \leqslant |k|^{|\Gamma|}$.

Proof. For any $\gamma \in \Gamma$, we have $B_{\geqslant\gamma}(0) = \bigcup_{\overline{a}\in k} a + B_{>\gamma}(0)$, where $a \in \mathcal{O}$, $\mathrm{res}(a) = \overline{a}$. Thus, we may recursively construct $f_\gamma : K/B_{>\gamma}(0) \to k$ in the following way: letting $(x_i + B_{>\gamma}(0))_{i\in I}$ enumerate $K/B_{>\gamma}(0)$, we choose $f_\gamma(x_i + B_{>\gamma}(0)) \in k$ such that if $x_i + B_{\geqslant\gamma}(0) \neq x_j + B_{\geqslant\gamma}(0)$ for all $j < i$, then $f_\gamma(x_i + B_{>\gamma}(0)) \neq f_\gamma(x_j + B_{>\gamma}(0))$. Then, we associate $a \in K$ with the function $g_a : \Gamma \to k$ defined by $g_a(\gamma) = f_\gamma(a + B_{>\gamma}(0))$. Note that if $a \neq b$, then $g_a(v(a-b)) \neq g_b(v(a-b))$; indeed $a - b \in B_{\geqslant v(a-b)}(0) \setminus B_{>v(a-b)}(0)$. $\square$

Corollary 17. *Any valued field K admits a maximal immediate extension K'. If $K \models$ ACVF, then $K' \models$ ACVF, and K' is an elemantary extension of K.*

Proof. Consider the set of all immediate extensions of K and apply Zorn's lemma to find K', thanks to the born obtained in proposition 16. Now if $K \models$ ACVF, its residue field is algebraically closed and its value group is divisible. Then K'^{alg} is an immediate extension of K', hence trivial, so K' must be algebraically closed. We have $K \preccurlyeq K'$ by completeness of $\mathrm{ACVF}_{\mathrm{ch}(K),\mathrm{ch}(k)}$. $\square$

References

[1] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. Definable sets in algebraically closed valued fields: elimination of imaginaries. *Journal fur die reine und angewandte Mathematik (Crelles Journal)*, 2006.

[2] Martin Hills. Model theory of valued fields. In Franziska Jahnke, Palacín Daniel, and Katrin Tent, editors, *Lectures in model theory*. European Mathematical Society, 2018.

[3] Will Johnson. On the proof of elimination of imaginaries in algebraically closed valued fields, 2014.